

DOMINANT REPRESENTING MEASURES AND RATIONAL APPROXIMATION

BY

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1. Let K be a compact set in the complex plane C . In 1951 Mergelyan proved that if $C \setminus K$ is connected, any continuous function on K analytic on the interior K^0 of K is uniformly approximable by polynomials in z . More generally, he later showed that if $C \setminus K$ has finitely many components, such functions are uniformly approximable by rational functions [M].

Following a functional analytic approach due to Bishop [B 3] of comparing the spaces of measures orthogonal to the two function algebras involved, the first of these results was derived in [GIW] via results valid for dirichlet algebras from a classical result of Walsh [Wa 1] and a well-known result of Bishop [B 2]. Since not only dirichlet algebras are involved in the second result of Mergelyan, the same approach was not then applicable.

Recently, however, one of the dirichlet algebra results used in [GIW], the abstract F. and M. Riesz theorem, was extended to its proper setting by P. R. Ahern [A]; this provided the origin for the present paper, which began as an attempt to obtain the more general Mergelyan result by the approach of [GIW]. In the special case in which the components of $C \setminus K$ have disjoint closures this is easily done⁽²⁾, but in general this requires answering some related questions, and leads to the interesting question of determining the dimension of the space of *real* measures on ∂K orthogonal to all rational functions with poles off K . The answer to this question has quite recently been obtained by P. R. Ahern and Donald Sarason [AS 2] as a consequence of their earlier study of H^p spaces for hypo-dirichlet (and slightly more general) algebras [AS 1], which had already given the Mergelyan result.

The main portion of this paper, §3, is devoted to an alternative proof of this result of Ahern and Sarason, and some extensions. We shall make use of two results of [AS 2] (Lemmas 1–3), both facts about harmonic measures resulting from the topology, independent of [AS 1] and also easily accessible; in general, however, our approach is quite different, depending more on the actual setting in the complex plane, and somewhat less on the global finite dimensionality of the real orthogonal measures implicit in the hypo-dirichlet situation, so that we obtain

Received by the editors September 29, 1966.

⁽¹⁾ Work supported in part by the National Science Foundation.

⁽²⁾ This and the preliminary results of this paper were announced in [Gl 2]; since the paper was completed many of the results of §2 have been extended considerably in [Gl 3], [GaGl].

some extensions to the case in which $C \setminus K$ has infinitely many components. These are very special, however, and in particular, our strongest results give more a means of constructing such K with the continuous analytic functions on K approximable by rational functions than a delineation of a specific class of sets for which such rational approximation holds (cf. (3.14), (3.15), and (3.19)).

§2 is devoted to the central notion of dominant representing measures, and gives an alternate proof of Ahern's F. and M. Riesz theorem and an extension of a result of Hoffman and Wermer crucial for all that follows, along with various corollaries. §4 is concerned with the application of results of the preceding section to interpolation, while the final section gives an illustration of the use of various preparatory results in obtaining approximation within the uniform closure on K of the rational functions with poles off K .

The author is indebted to Maynard G. Arsove for suggesting the proof of a key fact (Lemma 3.9) concerning harmonic measures, to John Wermer for pointing out some early errors, and to G. and L. Lumer, J. Ryff and W. B. Woelf for several helpful conversations. Our debt to [AS 2] will be clear.

In what follows $f\mu$ will denote the usual product of a function f and a measure μ , and, when convenient, $\mu(f)$ the integral $\int f d\mu$. For two nonnegative measures μ and λ , $\mu \ll \lambda$ will denote the absolute continuity of μ with respect to λ . For a set F , μ_F is the restriction of μ to F , and χ_F is its characteristic function.

Whenever a function algebra A is being viewed as a subalgebra of $C(X)$ for a particular X , A^\perp will denote the space of complex measures *on that* X orthogonal to A .

Finally, for want of a better term, we shall call a bounded domain U in \mathbb{C} *nicely connected* if it is simply connected and the (nontangential) boundary value extension of the Riemann map of $|z| < 1$ onto U is 1-1 on all but a subset of measure zero of the unit circle.

2. Dominant representing measures. Let X be a compact Hausdorff space, and A a closed separating subalgebra of $C(X)$ containing the constants. For $\phi \in \mathfrak{M} = \mathfrak{M}_A$, the space of multiplicative linear functionals on A (or spectrum of A), let $M_\phi = M_\phi(A)$ denote the (weak* compact, convex) set of all probability measures λ on X representing ϕ :

$$\phi(f) = \int f d\lambda = \lambda(f), \quad f \in A.$$

M_ϕ is nonvoid [We 3], and as a consequence $\operatorname{Re} \phi(f) = \int \operatorname{Re} f d\lambda$ is an (order preserving) function of $\operatorname{Re} f$ alone. This allows us to state the following (essentially familiar) consequence of the Hahn-Banach and Riesz representation theorems, which will be fundamental to our arguments.

LEMMA 2.1. For $u \in C^R(X)$,

$$(2.1) \quad \sup \{ \operatorname{Re} \phi(f) : f \in A, \operatorname{Re} f \leq u \} = \inf \{ \lambda(u) : \lambda \in M_\phi \}.$$

Proof. With f and u as above, $\operatorname{Re} \phi(f) = \int \operatorname{Re} f d\lambda \leq \int u d\lambda$ for any λ in M_ϕ , so the left side of (2.1) is bounded by the right. On the other hand, $\operatorname{Re} f \rightarrow \operatorname{Re} \phi(f)$ is a nonnegative linear functional, and by a version [D] of the Hahn-Banach theorem has an order-preserving extension to $C^R(X)$ with precisely the left side of (2.1) as its value at u .

But this extension is represented by a nonnegative measure λ on X , which must be a probability measure representing ϕ on A since $\lambda(\operatorname{Re} f) = \operatorname{Re} \phi(f)$, $f \in A$, and this implies $\lambda(1) = 1$ and $\lambda(f) = \phi(f)$, $f \in A$. So $\lambda \in M_\phi$ and $\lambda(u)$ is precisely the left side of (2.1), showing (2.1) holds.

DEFINITION 2.2. We call $\lambda \in M_\phi$ *dominant* (or a *dominant representing measure* for ϕ) if $M_\phi \subset L_1(\lambda)$; λ is *strongly dominant* if there is a constant k for which

$$\lambda' \leq k\lambda, \quad \text{all } \lambda' \in M_\phi.$$

Ahern's extension of the abstract F. and M. Riesz theorem (previously known when M_ϕ was a singleton) is to the case in which M_ϕ has a dominant representing measure. His result is basic to what we shall do, and we shall give a proof of the result (Theorem 2.4) based on (2.1). First, however, some remarks on dominant representing measures are in order.

REMARKS. (2.3a). A dominant representing measure for M_ϕ clearly exists when M_ϕ contains a norm dense sequence $\{\lambda_n\}$, e.g., $\sum_1^\infty 2^{-n}\lambda_n = \lambda$ is easily seen to be dominant. And when X is metrizable M_ϕ must be norm separable if a dominant representing measure λ exists since $L_1(\lambda)$ is then itself norm separable.

(2.3b). *More generally, a dominant λ exists whenever M_ϕ is contained in $L_1(\mu)$ for any measure μ .* This follows from a simple exhaustion argument, as follows. Let μ_λ be the absolutely continuous component of μ relative to $\lambda \in M_\phi$, and set $C = \sup \{\|\mu_\lambda\| : \lambda \in M_\phi\}$. Choose λ_n so that $\|\mu_{\lambda_n}\| \rightarrow C$, and then set $\lambda = \sum_1^\infty 2^{-n}\lambda_n$, so clearly $\|\mu_\lambda\| = C$. If $M_\phi \not\subset L_1(\lambda)$ we have a λ' in M_ϕ and $E \subset X$ with $\lambda(E) = 0$, $\lambda'(E) > 0$ whence $\mu(E) > 0$ and clearly $\|\mu_{(\lambda+\lambda')/2}\| > C$, our contradiction.

(2.3c). The difference of two elements of M_ϕ is a real measure orthogonal to A , and thus if the space of such real measures is norm separable, dominant representing measures exist for every $\phi \in \mathfrak{M}$. In particular, this is the case if $(\operatorname{Re} A)^\perp$, which amounts to the complex span of the real measures on X orthogonal to A , is *finite dimensional*. And in that case, *strongly dominant measures exist*. In fact, a strongly dominant $\lambda \in M_\phi$ exists whenever M_ϕ can be realized as a bounded convex body in some normed space. For if λ is interior to M_ϕ in that realization, since M_ϕ is bounded one has $\lambda + \varepsilon(\lambda - M_\phi) \subset M_\phi$ for a sufficiently small $\varepsilon > 0$, whence $(1 + \varepsilon)\lambda - \varepsilon\lambda' \geq 0$ for all λ' in M_ϕ .

(2.4). The principal consequence of the existence of a dominant representing measure is Ahern's F. and M. Riesz theorem [A], which follows from the following lemma, inspired in part by the approach of Forelli [F].

LEMMA 2.5 (AHERN [A]). *Suppose λ is a dominant representing measure for ϕ ,*

and E is a σ -compact set of λ -measure zero. Then there is a sequence $\{g_n\}$ in the unit ball of A with $g_n \rightarrow 0$ pointwise on E and $g_n \rightarrow 1$ a.e. λ .

Proof. Let K be a compact subset of E , and n a positive integer. Because K is compact and λ is regular, and $\lambda K = 0$, we can find a sequence $\{u_k\}$ in $C^R(X)$ with

- (i) $-n \leq u_k \leq 0$,
- (ii) $u_k = -n$ on K ,
- (iii) $\lambda(u_k) \rightarrow 0$.

Indeed we can clearly take $\{u_k\}$ nondecreasing, replacing u_k by $\sup \{u_j : j \leq k\}$.

Since $\{u_k\}$ is monotonic, (iii) implies $u_k \rightarrow 0$ a.e. λ , and since λ is dominant, $\lambda'(u_k) \rightarrow 0$ for each λ' in M_ϕ by monotone convergence. But now $f_k(\lambda') = \lambda'(u_k)$ defines a sequence $\{f_k\}$ in $C(M_\phi)$, where M_ϕ is given the weak* topology, and $f_k \nearrow 0$ pointwise on the compact space M_ϕ . By Dini's theorem convergence is uniform, and for k sufficiently large

$$-1/n < \inf \{\lambda'(u_k) : \lambda' \in M_\phi\}$$

and by (2.1) there is an f in A with

$$\operatorname{Re} f \leq u_k \leq 0, \quad -1/n < \operatorname{Re} \phi(f).$$

In particular $\operatorname{Re} f \leq -n$ on K by (ii), so that $|e^f| = e^{\operatorname{Re} f} \leq 1$ everywhere and $\leq e^{-n}$ on K , while $|\phi(e^f)| = |e^{\phi(f)}| = e^{\operatorname{Re} \phi(f)} > e^{-1/n}$. With $\xi = \operatorname{sgn} \phi(e^f)$, we thus have an element $g = \xi e^f$ in the unit ball of A with $1 \geq \phi(g) > e^{-1/n}$ and $|g| \leq e^{-n}$ on K .

Now E is the union of an increasing sequence K_n of compacta, and applying the preceding argument to n and $K = K_n$ we obtain a g_n in the unit ball of A with $|g_n| \leq e^{-n}$ on K_n (so $g_n \rightarrow 0$ on E) and $1 \geq \phi(g_n) > e^{-1/n}$. Since

$$\begin{aligned} 0 &\leq \int |1 - g_n|^2 d\lambda = 1 + \int |g_n|^2 d\lambda - 2 \operatorname{Re} \int g_n d\lambda \\ &= 1 + \int |g_n|^2 d\lambda - 2 \operatorname{Re} \phi(g_n) \leq 1 + 1 - 2 \operatorname{Re} \phi(g_n) \rightarrow 0, \end{aligned}$$

we have $g_n \rightarrow 1$ in $L_2(\lambda)$; passing to a subsequence, $g_n \rightarrow 1$ a.e. λ , as desired, and our proof is complete.

As an immediate consequence of Lemma 2.5 we have Ahern's F. and M. Riesz theorem.

THEOREM 2.6 [A]. Let $\mu \in A^\perp$ and let λ be a dominant representing measure for $\phi \in \mathfrak{M}$. Let μ_λ and μ'_λ be the absolutely continuous and singular components of μ with respect to λ . Then μ_λ and μ'_λ are orthogonal to A .

For the proof we need only note that μ'_λ is carried by a σ -compact set E of λ -measure zero, so that if $\{g_n\}$ is the sequence in the unit ball of A we obtain from (2.5) then $\|g_n \mu - \mu_\lambda\| = \|g_n \mu_\lambda + g_n \mu'_\lambda - \mu_\lambda\| \rightarrow 0$ by dominated convergence; since $g_n \mu \in A^\perp$ for each n , $\mu_\lambda \in A^\perp$, whence $\mu'_\lambda = \mu - \mu_\lambda \in A^\perp$.

In [G] Gleason introduced the notion of a "part" of \mathfrak{M} as an equivalence class in \mathfrak{M} under the equivalence relation that $\phi_1 \sim \phi_2$ if $\|\phi_1 - \phi_2\|_{A^*} < 2$ (where the functional norm is intended). An equivalent [H 1, 7.5] definition of this relation is that $\phi_1 \sim \phi_2$ iff $\hat{g}_n(\phi_1) \rightarrow 1$ whenever $\hat{g}_n(\phi_2) \rightarrow 1$, and vice versa, for any sequence $\{g_n\}$ in the unit ball of A . This yields the next consequence of 2.5.

COROLLARY 2.7. *If ϕ has a dominant representing measure λ then any ϕ_1 in the same Gleason part has a dominant representing measure equivalent to (i.e., mutually absolutely continuous with) λ .*

Let $\lambda_1 \in M_{\phi_1}$ and let E be a compact set with $\lambda(E) = 0$. By 2.5 we have a sequence $\{g_n\}$ in the unit ball of A with $g_n \rightarrow 0$ on E and $g_n \rightarrow 1$ a.e. λ , so that $\hat{g}_n(\phi) \rightarrow 1$. Thus $\lambda_1(g_n) = \hat{g}_n(\phi_1) \rightarrow 1$, and since $\limsup |\lambda_1(g_n)| \leq 1 - \lambda_1(E)$, we conclude that $\lambda_1(E) = 0$, and therefore that $\lambda_1 \ll \lambda$. As noted in (2.3b), $M_{\phi_1} \subset L_1(\lambda)$ implies ϕ_1 has a dominant representing measure λ_1 , and of course $\lambda_1 \ll \lambda$. By symmetry $\lambda \ll \lambda_1$, and we are done.

If $\phi_1, \phi_2 \in \mathfrak{M}$ have representing measures λ_1, λ_2 which are not mutually singular, then $\|\phi_1 - \phi_2\|_{A^*} \leq \|\lambda_1 - \lambda_2\| < 2$, so that $\phi_1 \sim \phi_2$. Thus *any two dominant representing measures are either equivalent or mutually singular*, depending on whether the corresponding homomorphisms are in the same or distinct parts.

DEFINITIONS. (1) *A measure σ orthogonal to A is completely singular if $|\sigma|$ and each representing measure are mutually singular.*

(2) *For convenience we call a part trivial if it consists of a peak point.*

We of course have a dominant representing measure for such a point, but no nonzero orthogonal measures are absolutely continuous with respect to the dominant one, and for this reason trivial parts need not appear in our next result, which follows exactly as in [GIW] by virtue of 2.6.

THEOREM 2.8. *Suppose each nontrivial part P_v of \mathfrak{M} contains an element with a dominant representing measure λ_v . Then for any μ in A^\perp we have*

$$(2.2) \quad \mu = \sum_v \mu_{\lambda_v} + \sigma,$$

where σ is a completely singular orthogonal measure, μ_{λ_v} is the absolutely continuous part of μ relative to λ_v , the summands on the right are mutually singular and orthogonal to A , and the (actually countable) sum converges in norm.

Setting $A_{\lambda_v}^\perp = \{\mu_{\lambda_v} : \mu \in A^\perp\}$, which coincides with $\{\mu \in A^\perp : \mu \ll \lambda_v\}$ by 2.6, and can be viewed as a closed subspace of $L^1(\lambda_v)$, we can write (2.2) as

$$(2.3) \quad A^\perp = \sum_v A_{\lambda_v}^\perp + S,$$

where S is the closed space of completely singular orthogonal measures and the first summand represents all norm convergent series. In order to further identify

the components A_λ^\perp we need strongly dominant representing measures (cf. (2.2)) to obtain the following extension of a result⁽³⁾ of Hoffman and Wermer [We 5]. First we recall the

DEFINITION. $H^p(\lambda) = H^p(A, \lambda)$ denotes the closure in $L^p(\lambda)$, $p < \infty$, of A , and $H^\infty(\lambda) = L^\infty(\lambda) \cap H^2(\lambda)$. Finally $H_0^p(\lambda) = \{f \in H^p(\lambda) : \int f d\lambda = 0\}$.

THEOREM 2.9. Suppose λ is a strongly dominant representing measure for ϕ , and $h \in L^\infty(\lambda) \cap H^2(\lambda) = H^\infty(A)$. Then there is a sequence $\{f_n\}$ in A with $\|f_n\| \leq \|h\|_\infty$, which converges to h a.e. λ .

Thus $h \in L^\infty(\lambda) \cap H^2(\lambda)$ implies h lies in the weak* closure of A in $L^\infty(\lambda)$.

Proof (following [We 5]). Let $\{g_n\}$ be a sequence in A converging to h in $L^2(\lambda)$. Replacing $\{g_n\}$ by a subsequence we can assume $g_n \rightarrow h$ a.e. λ , and without loss of generality we can assume $\|h\|_\infty = 1$.

Let $E_n = \{x \in X : |g_n(x)| \geq 1\}$. We assert that $\int_{E_n} \log |g_n| d\lambda \rightarrow 0$. Indeed for $\varepsilon > 0$ if $G_n = \{x \in X : |g_n(x)| \geq 1 + \varepsilon\}$ then $\lambda(G_n) \rightarrow 0$ and

$$\begin{aligned} \int_{G_n} \log |g_n| d\lambda &\leq \int_{G_n} |g_n| d\lambda \leq \int_{G_n} |g_n - h| + |h| d\lambda \\ &\leq \|g_n - h\|_2 \cdot 1 + \lambda(G_n) \rightarrow 0. \end{aligned}$$

So $\int_{G_n} \log |g_n| d\lambda < \varepsilon$ for n large, and

$$\int_{E_n} \log |g_n| d\lambda \leq \int_{G_n} + \int_{E_n \setminus G_n} < \varepsilon + \log(1 + \varepsilon)$$

since $|g_n| < 1 + \varepsilon$ off G_n . Our assertion is now clear.

Now $u_n = -\log^+ |g_n| \in C^R(X)$ vanishes off E_n , $u_n \leq 0$, and $0 \leq \varepsilon_n = -\int u_n d\lambda = \int \log^+ |g_n| d\lambda = \int_{E_n} \log |g_n| d\lambda \rightarrow 0$. Since λ is strongly dominant $k\lambda \geq \lambda'$ for all λ' in M_ϕ , whence

$$0 \geq \int u_n d\lambda' \geq \frac{1}{k} \int u_n d\lambda = -\frac{\varepsilon_n}{k},$$

u_n being nonpositive. So

$$\inf \{\lambda'(u_n) : \lambda' \in M_\phi\} \geq -\varepsilon_n/k$$

and by (2.1) we have an h_n in A with $\operatorname{Re} h_n \leq u_n \leq 0$, and $\operatorname{Re} \phi(h_n) > -\varepsilon_n/k - 1/n$. Thus $|e^{h_n}| \leq 1$, and in fact $|g_n e^{h_n}| = |g_n| e^{\operatorname{Re} h_n} \leq |g_n| e^{-\log^+ |g_n|} \leq 1$ while $1 \geq |\phi(e^{h_n})| = e^{\operatorname{Re} \phi(h_n)} > e^{-\varepsilon_n/k - 1/n}$ which tends to 1 as $n \rightarrow \infty$. So if we set

$$h'_n = e^{h_n} \cdot \overline{\operatorname{sgn} \phi(e^{h_n})}$$

we obtain a sequence $\{h'_n\}$ in the unit ball of A with $h'_n \rightarrow 1$ a.e. λ , exactly as in the last part of the proof of 2.5, and $|g_n h'_n| \leq 1$. Now we need only set $f_n = g_n h'_n$ to get our sequence $\{f_n\}$ in A with $\|f_n\| \leq \|h\|_\infty$, $f_n \rightarrow h$ a.e. λ .

⁽³⁾ The version of this result suggested (with no mention of strong dominance) in [Gl 2] is erroneous.

Finally, by dominated convergence $f_n \rightarrow h$ weak* in $L^\infty(\lambda)$, completing our proof.

We can now identify A_λ^\perp when λ is strongly dominant. Recall first that $(\operatorname{Re} A)^\perp$, the measures on X orthogonal to $\operatorname{Re} A$, is spanned by real measures, which, since $\operatorname{Re} A = \operatorname{Im} A$, are orthogonal to A . Thus $(\operatorname{Re} A)^\perp$ is just the complex span of the real measures in A^\perp . We again set $(\operatorname{Re} A)_\lambda^\perp = \{\mu_\lambda : \mu \in (\operatorname{Re} A)^\perp\}$, which is the span in $L_1(\lambda) \cdot \lambda$ of its real elements orthogonal to A .

COROLLARY 2.10. *If λ is strongly dominant representing measure for ϕ then A_λ^\perp is the closure in the measure norm of $(\operatorname{Re} A)_\lambda^\perp + \phi^{-1}(0) \cdot \lambda$. In particular, if $(\operatorname{Re} A)_\lambda^\perp$ is finite dimensional,*

$$A_\lambda^\perp = (\operatorname{Re} A)_\lambda^\perp + (\phi^{-1}(0) \cdot \lambda)^\perp = (\operatorname{Re} A)_\lambda^\perp + H_0^1(\lambda) \cdot \lambda.$$

In order to prove 2.10 we need an observation on $L^2(\lambda)$. As is easily seen λ is multiplicative on $H^2(\lambda)$ (by Schwartz, $(f, g) \rightarrow \int fg \, d\lambda$ extends continuously from $A \times A$ to $H^2 \times A$ to $H^2 \times H^2$); so we have $H_0^2(\lambda) = \{f \in H^2(\lambda) : \int f \, d\lambda = 0\}$ orthogonal in $L^2(\lambda)$ to

$$\overline{H^2(\lambda)} : f \in H_0^2(\lambda), \quad g \in H^2(\lambda)$$

imply $\int fg \, d\lambda = 0$. Moreover a standard elementary argument shows $H_0^2(\lambda)$ is the closure of $\phi^{-1}(0)$ in $L^2(\lambda)$. Thus we have the orthogonal decomposition

$$L^2(\lambda) = H^2(\lambda) \oplus \overline{H_0^2(\lambda)} \oplus E,$$

where E is orthogonal to

$$A + \overline{\phi^{-1}(0)} = A + \bar{A},$$

hence to $\operatorname{Re} A$: so E lies in the span of the real elements of $L^2(\lambda)$ orthogonal to A , and clearly coincides with that span. Now we may write, with obvious but clumsy notation

$$(2.4) \quad L^2(\lambda) = H^2(\lambda) \oplus \overline{H_0^2(\lambda)} \oplus (\operatorname{Re} A)_\lambda^\perp.$$

To proceed to the proof of 2.10, we have $A_\lambda^\perp \supset (\operatorname{Re} A)_\lambda^\perp + \phi^{-1}(0) \cdot \lambda$ clearly, so we need only see any h in $L^\infty(\lambda)$ orthogonal to the smaller set is orthogonal to A_λ^\perp . But such an h lies in $L^2(\lambda)$ and in terms of L^2 orthogonality \bar{h} is orthogonal to $(\operatorname{Re} A)_\lambda^\perp + \phi^{-1}(0)$, so h is orthogonal to $(\operatorname{Re} A)_\lambda^\perp + \phi^{-1}(0) \cdot \lambda$, hence lies in $H^2(\lambda)$ by (2.4). So by 2.9 we have $\{f_n\}$ in A converging weak* to h in $L^\infty(\lambda)$, and thus for any μ in $A_\lambda^\perp \subset L_1(\lambda) \cdot \lambda$, $0 = \int f_n \, d\mu \rightarrow \int h \, d\mu$, and $h \perp A_\lambda^\perp$ as desired.

For the final assertion of 2.10 we need only note that by [D, p. 14] $(\operatorname{Re} A)_\lambda^\perp + (\phi^{-1}(0) \cdot \lambda)^\perp$ is closed, while $H_0^1(\lambda) = (\phi^{-1}(0) \cdot \lambda)^\perp$ as we noted earlier.

We shall want to apply the preceding results to the following question. Suppose we have two subalgebras $A \subset B$ of $C(X)$ for which $\mathfrak{M}_A = \mathfrak{M}_B$. Under what conditions can we assert $A = B$? We are of course limited to those situations in which dominant or strongly dominant representing measures exist, and in the actual cases we consider no completely singular measures exist. One case to which our results apply can be summed up as follows.

COROLLARY 2.11. *Suppose $A \subset B \subset C(X)$ have the same space \mathfrak{M} of multiplicative linear functionals and $(\operatorname{Re} A)^\perp = (\operatorname{Re} B)^\perp$ is finite dimensional. Then if A has no completely singular orthogonal measures, $A = B$.*

We could alternatively assume the defects in $C^R(X)$ of $(\operatorname{Re} A)^\perp$ and $(\operatorname{Re} B)^\perp$ be finite and equal, and more generally that the completely singular orthogonal measures for A and B simply coincide—but we have no application of this sort in mind. In order to see 2.11 note that for every multiplicative measure λ_1 on A the corresponding homomorphism arises from one of B by hypothesis, represented on $B \supset A$ by a measure λ_2 multiplicative on B . So $\lambda_1 - \lambda_2 \in (\operatorname{Re} A)^\perp = (\operatorname{Re} B)^\perp$, and λ_1 is multiplicative on B as well. Thus the multiplicative measures for the two algebras coincide.

Since $(\operatorname{Re} A)^\perp = (\operatorname{Re} B)^\perp$ is finite dimensional, strongly dominant measures exist by (2.3c), and these clearly coincide for both algebras by their definition. Since parts in $\mathfrak{M}_A = \mathfrak{M}_B$ for each algebra are determined by the equivalence or mutual singularity of dominant representing measures (2.7 and below), the parts coincide for both algebras (a fact which can be seen in various other ways). Moreover, a completely singular measure orthogonal to B is orthogonal to A , so must vanish. Thus, we have a collection $\{\lambda_\nu\}$ of strongly dominant representing measures, one for each part, and by Theorem 2.8 and (2.3) $A^\perp = \sum A_{\lambda_\nu}^\perp$, $B^\perp = \sum B_{\lambda_\nu}^\perp$. Hence, by Hahn-Banach we need only see $A_\lambda^\perp = B_\lambda^\perp$ for each strongly dominant λ to complete our proof.

Now (2.4) applies to each algebra:

$$\begin{aligned} (2.5) \quad L^2(\lambda) &= H^2(A, \lambda) \oplus \overline{H_0^2(A, \lambda)} \oplus (\operatorname{Re} A)_{\frac{1}{2}}^\perp \\ &= H^2(B, \lambda) \oplus \overline{H_0^2(B, \lambda)} \oplus (\operatorname{Re} B)_{\frac{1}{2}}^\perp, \end{aligned}$$

and since

$$A \subset B, \quad H^2(A, \lambda) \subset H^2(B, \lambda), \quad \overline{H_0^2(A, \lambda)} \subset \overline{H_0^2(B, \lambda)};$$

and $(\operatorname{Re} A)_{\frac{1}{2}}^\perp \subset (\operatorname{Re} B)_{\frac{1}{2}}^\perp$ as well since $(\operatorname{Re} A)_{\frac{1}{2}}^\perp \cdot \lambda = (\operatorname{Re} A)^\perp \cap L_2(\lambda) \cdot \lambda = (\operatorname{Re} B)^\perp \cap L_2(\lambda) \cdot \lambda = (\operatorname{Re} B)_{\frac{1}{2}}^\perp \cdot \lambda$. So we have equality in each case, in particular

$$(2.6) \quad H^2(A, \lambda) = H^2(B, \lambda).$$

We have $B^\perp \subset A^\perp$, so $B_\lambda^\perp \subset A_\lambda^\perp (\subset L_1(\lambda) \cdot \lambda)$. To see we have equality, let $h \in L^\infty(\lambda)$ be orthogonal to B_λ^\perp . Then h is orthogonal to $(H_0^2(B, \lambda) + (\operatorname{Re} B)_{\frac{1}{2}}^\perp) \cdot \lambda \subset B_\lambda^\perp$, hence by (2.5) $h \in H^2(B, \lambda)$. But by (2.6) $h \in H^2(A, \lambda)$, and so by Theorem 2.9 we have h in the weak* closure of A in $L^\infty(\lambda)$; thus $h \perp A_\lambda^\perp$ and $A_\lambda^\perp \subset B_\lambda^\perp$, so $A_\lambda^\perp = B_\lambda^\perp$. As we saw this completes our proof.

The main part of this argument applies more generally.

COROLLARY 2.12. *Let $A \subset B$ be closed subalgebras of $C(X)$ and let λ be a measure multiplicative on B which is strongly dominant for the induced functional on A . Then $A_\lambda^\perp = B_\lambda^\perp$ iff $H^2(A, \lambda) = H^2(B, \lambda)$ (or equivalently by (2.5), iff $(\operatorname{Re} A)_{\frac{1}{2}}^\perp = (\operatorname{Re} B)_{\frac{1}{2}}^\perp$ in the notation used there).*

Suppose the H^2 spaces coincide. We have $B_\lambda^\perp \subset A_\lambda^\perp$, and if $h \in L^\infty(\lambda)$ is orthogonal to B_λ^\perp we obtain $h \in H^2(B, \lambda)$ as before. (This uses the multiplicativity of λ on B ; otherwise the decomposition in (2.5) need not be orthogonal.) Since then $h \in H^2(A, \lambda)$ we again conclude that $h \perp A_\lambda^\perp$ as in the previous proof. So $A_\lambda^\perp = B_\lambda^\perp$.

Conversely, if $A_\lambda^\perp = B_\lambda^\perp$ their intersections with $L^2(\lambda) \cdot \lambda$ coincide, so that $H_0^2(A, \lambda) + (\text{Re } A)_2^\perp = H_0^2(B, \lambda) + (\text{Re } B)_2^\perp$, whence $H^2(A, \lambda) = H^2(B, \lambda)$ by (2.5).

COROLLARY 2.13. *Suppose $A \subset B$ as before, and share the same set of multiplicative measures. Suppose moreover that on A distinct multiplicative measures induce distinct functionals (so M_ϕ is a singleton for each $\phi \in \mathfrak{M}$, or representing measures are "unique"). Then $\sum A_\lambda^\perp = \sum B_\lambda^\perp$ in the notation of (2.3), and $A = B$ if the completely singular orthogonal measures for the two algebras coincide.*

Each multiplicative measure λ is of course strongly dominant for the corresponding homomorphism ϕ on A or B , so that (2.3) holds. And as is well known [L, H1], uniqueness of the representing measure λ for ϕ implies

$$\begin{aligned} L^2(\lambda) &= H^2(A, \lambda) \oplus \overline{H_0^2(A, \lambda)} \\ &= H^2(B, \lambda) \oplus \overline{H_0^2(B, \lambda)} \end{aligned}$$

so that $H^2(A, \lambda) = H^2(B, \lambda)$. So by 2.12, $A_\lambda^\perp = B_\lambda^\perp$, yielding our first assertion; the second follows from 2.8.

We should note a related consequence of the preceding.

COROLLARY 2.14. *Let A be a closed subalgebra of $C(X)$ and suppose that in each nontrivial part we have some ϕ with a strongly dominant representing measure λ , with Λ the collection of all such λ . Suppose there are no completely singular orthogonal measures. Then*

$$A = \{f \in C(X) : f \in H^2(\lambda) \text{ for all } \lambda \text{ in } \Lambda\}.$$

The right side is an algebra (by 2.9, or directly) containing A , but all we need note is that if one of its elements f is not in A there is a μ in A^\perp , hence one in A_λ^\perp for some $\lambda \in \Lambda$, not orthogonal to f . Since $f \in H^2(\lambda) \cap L^\infty(\lambda) = H^\infty(\lambda)$, by 2.9 $a_n \rightarrow f$ weak* for some sequence $\{a_n\}$ in A , so $\mu(f) = \lim \mu(a_n) = 0$.

We should note that $H^\infty(\lambda)$ (or the set in 2.14) is always an algebra. For $f \in H^\infty(\lambda)$ implies $Af \subset H^2(\lambda)$, whence $f \cdot H^2(\lambda) \subset H^2(\lambda)$ since f is bounded, so

$$fH^\infty \subset H^2 \cap L^\infty = H^\infty.$$

3. Rational approximation in the plane. Let K be a compact subset in the complex plane C , and let $R(K)$ denote the closure in $C(K)$ of the rational functions with poles off K , $A(K)$ the larger closed subalgebra of $C(K)$ of functions analytic on K^0 , the interior of K . For each of these algebras evaluation at the elements of K provides the elements of \mathfrak{M} [Ar], and we may identify \mathfrak{M} and K . And in each

case ∂K , the topological boundary, provides the Šilov boundary of the algebra, which we shall *always* take as the X of §2. It will be convenient to speak sometimes of $R(K)$ and $A(K)$ as subalgebras of $C(\partial K)$ (meaning of course the isomorphic algebras of restrictions), and in this section we shall *always consider orthogonal measures carried only by ∂K* . We shall consider K as a subset of the Riemann sphere S so that the point at infinity will play no special rôle, but we shall always want it to lie outside K .

Fundamental to our considerations will be a theorem of Walsh [Wa 1] which can be stated on S as follows: if U is a nonvoid connected open subset of S and $K = S \setminus U$, then $R(K)$, as an algebra on ∂K , is dirichlet⁽⁴⁾. This implies [We 3] that each $z \in \partial K$ is a peak point for some f in $R(K)$ (so, by definition, ∂K is the *minimal boundary* [B 2] for $R(K)$). Thus if K is a compact proper subset of S whose boundary is just the union of the boundaries of the components of $S \setminus K$, then each z in ∂K is a peak point for $R(K)$, hence for $A(K)$, so ∂K is the minimal boundary for $A(K)$. In all that follows we shall consider only K for which ∂K is the minimal boundary for $A(K)$, in particular those for which $S \setminus K$ has finitely many components.

The fact that ∂K is the minimal boundary for $A(K)$ implies⁽⁵⁾ [T] that any continuous function f on ∂K has a continuous extension \hat{f} to K harmonic on K^0 (i.e., the dirichlet problem is solvable). This allows a simple definition of harmonic measures, and a brief introduction of these is perhaps in order. The map $f \rightarrow \hat{f}$ of $C^R(\partial K)$ into continuous (real-valued) harmonic functions is of course order preserving, so for $z \in K$, $f \rightarrow \hat{f}(z)$ is a nonnegative linear functional, 1 at 1; the corresponding (unique) probability measure λ_z ($= \delta_z$ if $z \in \partial K$) on ∂K is the harmonic measure for z , and as is easily seen, is carried (for $z \in K^0$) by the boundary of the component of K^0 containing z (take $f \in C^R(\partial K)$ peaking on that part of ∂K). Each λ_z is a multiplicative measure on $A(K)$, since $f(z) = \int f d\lambda_z$ for $f \in A(K)$, f being continuous and harmonic. Moreover $z \rightarrow \lambda_z$ is weak* continuous (as a map into $C^R(\partial K)^*$) since each $\hat{f} \in C(K)$.

Because of this continuity if μ is any measure carried by K we can form the measure on ∂K given by the weak* convergent integral

$$(3.1) \quad \int \lambda_z \mu(dz);$$

doing so is “sweeping the measure μ to ∂K ”, and we have the swept measure (3.1) and μ agreeing on $A(K)$, since $\int f(z) d\mu = \int \lambda_z(f) \mu(dz)$, $f \in A(K)$. One use of sweeping is to see that

3.1. *If we wish to show the real measures μ on ∂K orthogonal to $R(K)$ have an at most n -dimensional span we need only prove this for some sequence $\{K_\nu\}$ of larger compacta which decrease to K .*

(4) I.e., $\text{Re } R(K)$ is dense in $C^R(\partial K)$. A more direct “functional analytic” proof of Walsh’s theorem is the content of the first three lemmas of [C].

(5) We shall require a minimal acquaintance with basic facts concerning harmonic functions, as covered for example in pp. 1–11 of [T].

For if λ_z^ν is the harmonic measure for z relative to K , then $\mu_\nu = \int \lambda_z^\nu \mu(dz)$ is a real measure on ∂K_ν orthogonal to $R(K_\nu)$, and $\mu_\nu \rightarrow \mu$ weak* since any weak* cluster point of λ_z^ν will be a measure on ∂K representing z on rational functions, hence coinciding with δ_z (since ∂K is assumed the minimal boundary of $A(K)$). So $\lambda_z^\nu \rightarrow \delta_z$ weak*, whence $\mu_\nu \rightarrow \mu$ weak* by dominated convergence. Thus if the span of the swept measures μ_ν on ∂K_ν is at most n -dimensional for each ν , the same is true of the span of the μ .

Two other properties of harmonic measures will be useful. First $z \rightarrow \lambda_z$ is *norm continuous* on K^0 . This is evident if K is a closed disc D (where harmonic measure is the Poisson kernel multiplying Lebesgue measure on the boundary), and if D is a closed disc in K^0 then for z in D sweeping λ_z^D out to ∂K yields λ_z : $\lambda_z = \int \lambda_w \lambda_z^D(dw)$. So if $\|\lambda_z^D - \lambda_{z'}^D\| < \varepsilon$, $\|\lambda_z - \lambda_{z'}\| = \|\int \lambda_w(\lambda_z^D - \lambda_{z'}^D)(dw)\| < \varepsilon$. The final fact we need can also be obtained by sweeping (several) λ_z^D out to ∂K : if z_1, z_2 lie in the same component of K^0 , λ_{z_1} and λ_{z_2} are boundedly equivalent measures (i.e., with bounded Radon-Nikodym derivatives).

As indicated earlier, our attack on Mergelyan's rational approximation theorem using the results of §2 follows from an evaluation of the defect, in $C^R(\partial K)$, of $(\operatorname{Re} R(K))^-$, and of $(\operatorname{Re} A(K))^-$, or alternatively of the dimensions of the spans of the real orthogonal measures carried by ∂K . For convenience we shall call the defect of $(\operatorname{Re} R(K))^-$ the *rational defect*, and use *analytic defect* when the algebra is $A(K)$. The determination of the rational defect when $S \setminus K$ has finitely many components will be one of the main results of this section, reproducing the Ahern-Sarason result in Theorem 3.13: the defect is the sum of the connectivities of the components of K^0 . However we feel the main interest in our proof, which is long and involved, lies more in further applications of the methods used; consequently we have interposed many remarks only needed after 3.13, and the reader might be well advised to eliminate certain portions of the text on first reading. Thus one can arrive at the Ahern-Sarason result by reading only Lemmas 3.2, 3.4, 3.5, the first part of Lemma 3.6, and then 3.9 through 3.13 (omitting all parenthetical remarks as well).

When $S \setminus K$ has $n+1$ components H_0, \dots, H_n a second theorem of Walsh [Wa 2] shows the rational defect is at most n (the first cited theorem corresponds to the case $n=0$), so that strongly dominant representing measures exist for either algebra and each $z \in K$ (2.3c). When the H_i have disjoint boundaries F_i the characteristic functions $\chi_{F_1}, \dots, \chi_{F_n}$ are independent elements of $C^R(\partial K)$ not approximable by⁽⁶⁾

(⁶) If $f \in A(K)$ and $\operatorname{Re} f$ approximates χ_{F_1} well, then $f(K)$ lies in two disjoint vertical strips about $\operatorname{Re} z=0$, $\operatorname{Re} z=1$, and applying to f an analytic function $=0$ on the left strip and $=1$ on the right we obtain an element g in $A(K)$ which has $g|_{\partial K} = \chi_{F_1}$. But then $g(K) = \{0, 1\}$, and g provides a separation $K = A \cup B$ of K where $g(A)=1$, $g(B)=0$, and $F_1 \subset A$. It is easy to see $\partial A = F_1 = \partial H_1$, and since each $z \in \partial H_1$ has a neighborhood lying wholly within $A \cup H_1$ we can extend g to $K_1 = K \cup H_1$ by setting $g=1$ on H_1 to obtain $g \in A(K_1)$ which is $\equiv 0$ on $\partial K_1 = \bigcup_{j \neq 1} F_j$, and yet is nonzero. (We can also see the analytic defect is at least n by exhibiting n real independent orthogonal measures, as in 3.13 below.)

$\text{Re } A(K)$. Thus the analytic defect is at least n , whence both defects coincide, $(\text{Re } R(K))^\perp = (\text{Re } A(K))^\perp$, and 2.11 applies once we see (in 3.6) there are no completely singular orthogonal measures.

Actually the first cited theorem of Walsh will suffice for our purposes, because of the following lemma which allows us to combine certain dirichlet algebras. Besides later utility it also has applicability to subalgebras of $A(K)$ other than $R(K)$.

LEMMA 3.2. Suppose $K = K_0 \cap K_1$, and we are given closed subalgebras A_i of $C(K_i)$ with Šilov boundaries $\partial_i = \partial K_i$, $i=0, 1$ for which each f in A_i is analytic on $K_i \setminus \partial_i$. Suppose A_0 is dirichlet on ∂_0 , while $(\text{Re } A_1)^\perp$ (on ∂_1) is n -dimensional, and ∂_1 lies on one component U of the interior of K_0 . Then if A is the closed subalgebra of $C(K)$ generated by A_0 and A_1 , $\partial = \partial_0 \cup \partial_1$ is the Šilov boundary of A and

- (i) $(\text{Re } A)^\perp$ (on ∂ of course) is at most $(n+1)$ -dimensional⁽⁷⁾.
- (ii) For $\mu \in (\text{Re } A)^\perp$, μ_{∂_0} is absolutely continuous with respect to λ_z^U , for any $z \in U$.

[The same sort of argument works when ∂_i is a larger subset of K_i . So, for example, we could let $K_0 = K_1 = S$, and let ∂_0 and ∂_1 be disjoint replicas of Wermer's arc for which $A_i = \{f \in C(S) : f \text{ analytic on } S \setminus \partial_i\}$ is dirichlet on ∂_i [We 1]. Then the algebra A has an at most one-dimensional space of real orthogonal measures on $\partial = \partial_0 \cup \partial_1$; replacing A_1 by this algebra and introducing another arc ∂_2 disjoint from $\partial_0 \cup \partial_1$ the argument applies again. So if $\partial_0, \dots, \partial_n$ are a pairwise disjoint collection of such arcs the algebra we finally generate, and so that larger algebra of f in $C(S)$ analytic off $\partial = \bigcup_{i=0}^n \partial_i$, has an at most n -dimensional space of real orthogonal measures on its Šilov boundary (and one can easily see these are all boundedly absolutely continuous with respect to the harmonic measure for $S \setminus \partial$, as in 3.13 below). Indeed the dimension is precisely n , as one can see as in footnote 6. Finally, suppose $\partial_0, \partial_1, \dots, \partial_n$ are all interior to the unit disc D ; then the algebra A of $f \in C(D)$ analytic on $D^0 \setminus \bigcup_{i=0}^n \partial_i$ has an $(n+1)$ -dimensional space of real orthogonal measures.]

To begin the proof of 3.2, let ν_1, \dots, ν_n be the real measures on ∂_1 spanning $(\text{Re } A_1)^\perp$, and let μ be a real measure on $\partial = \partial_0 \cup \partial_1$ orthogonal to A . Let $\mu_i = \mu_{\partial_i}$. Since $\partial K_i = \partial_i$, ∂_i carries the harmonic measure λ_z^i of $z \in K_i$ relative to K_i . Sweeping μ_1 to ∂_0 we have $\mu_0 + \int \lambda_z^0 \mu_1(dz)$ a real measure on ∂_0 orthogonal to A_0 since $f \in A_0 \subset A$ implies $\mu_0(f) + \int \lambda_z^0(f) \mu_1(dz) = \mu_0(f) + \int f(z) \mu_1(dz) = (\mu_0 + \mu_1)(f) = 0$. Since A_0 is dirichlet on ∂_0 ,

$$(3.2) \quad \mu_0 = - \int \lambda_z^0 \mu_1(dz).$$

Similarly $\mu_1 + \int \lambda_w^1 \mu_0(dw)$ is a real measure on ∂_1 orthogonal to A_1 , whence we have real c_i with

$$(3.3) \quad \mu_1 + \int \lambda_w^1 \mu_0(dw) = - \sum_{i=1}^n c_i \nu_i.$$

⁽⁷⁾ Since $(\text{Re } A)^\perp$ is spanned by real measures it is sufficient to show its real elements form a real space of dimension $\leq n+1$.

Since $\partial_1 \subset U$, $\lambda_z^0 = \lambda_z^U$ for $z \in \partial_1$ and thus (3.2) implies (ii), all measures λ_z^0 , $z \in U$, being equivalent. If we let X denote the closed carrier of any of the equivalent measures λ_z^0 , $z \in \partial_1$, then $X \subset \partial U \subset \partial K_0 \subset \partial_0$ carries μ_0 by (3.2).

Combining (3.2) and (3.3)

$$(3.4) \quad \mu_0 = \iint \lambda_z^0 \lambda_w^1(dz) \mu^0(dw) + \sum c_i \int \lambda_z^0 \nu_i(dz),$$

or, setting $\eta_w = \int \lambda_z^0 \lambda_w^1(dz)$, a probability measure on X , and $\sigma_i = \int \lambda_z^0 \nu_i(dz)$, again a measure on X ,

$$(3.5) \quad \mu_0 = \int \eta_w \mu_0(dw) + \sum_{i=1}^n c_i \sigma_i.$$

Now set $Tf(w) = \eta_w(f)$, $f \in C^R(X)$. This defines a nonnegative bounded linear operator on $C^R(X)$ for which (3.5) says

$$(3.6) \quad (I - T^*)\mu_0 = \sum_{i=1}^n c_i \sigma_i.$$

As we observed $w \rightarrow \lambda_w^1$ is strongly continuous on K_1^0 , and since ∂_0 is disjoint from $U \supset \partial_1 \supset \partial K_1$ while $\partial_0 \subset K \subset K_1$, $\partial_0 \subset K_1^0$ so that $w \rightarrow \lambda_w^1$ in particular is strongly continuous on ∂_0 . This implies T takes the unit ball of $C^R(X)$ into an equicontinuous set:

$$|f(w) - f(w')| = \left| \int \lambda_z^0(f)(\lambda_w^1 - \lambda_{w'}^1)(dz) \right| \leq \|f\| \|\lambda_w^1 - \lambda_{w'}^1\|.$$

Hence T is a compact operator.

Since $T1 = 1$, 1 is an eigenvalue of T ; in fact the nullity of $(I - T)$, and of $(I - T)^2$, is spanned by the constant function 1. To see this note that η_w has carrier precisely X so that if $f(w) = \eta_w(f)$ for all w in X for some $f \in C^R(X)$, selecting w so as to maximize $f(w)$ shows f assumes this value on all of X . Again if $(I - T)^2 f = 0$, $(I - T)f$ is a constant, so $f(w) - \eta_w(f) = c$. If $c > 0$, (respectively $c < 0$) selecting w so that $f(w)$ is a minimum (respectively maximum) leads to a contradiction, so we conclude $c = 0$ and have f a constant as before.

So 1 is a simple eigenvalue of T , and so, by the Riesz theory of compact operators [RN], of T^* . Thus we have a one-dimensional nullity for $I - T^*$ and for $(I - T^*)^2 = (I - T)^{2*}$, so that $(I - T^*)$ is invertible on the range of $I - T^*$, and the space of measures on X is the direct sum of the nullity and range [RN].

If $(I - T^*)\tau_0 = 0$, $\tau_0 \neq 0$ then since the space of measures on X is the direct sum of nullity and range, we have a real c_0 for which $\mu_0 - c_0\tau_0$ is in the range of $I - T^*$, and similarly real k_i for which $\sigma_i - k_i\tau_0$ is in that range. By (3.6)

$$\begin{aligned} (I - T^*)\mu_0 &= (I - T^*)(\mu_0 - c_0\tau_0) = \sum c_i \sigma_i \\ &= \sum c_i(\sigma_i - k_i\tau_0) + \left(\sum c_i k_i\right)\tau_0 \end{aligned}$$

and since all but the last term lie in the range of $I - T^*$ we conclude $\sum c_i k_i \tau_0 = 0$, in view of our direct sum. Applying the inverse S of the restriction of $I - T^*$ to its range we have

$$\mu_0 - c_0 \tau_0 = \sum_{i=1}^n c_i \tau_i, \quad \tau_i = S(\sigma_i - k_i \tau_0),$$

and so by (3.4)

$$\begin{aligned} \mu_1 &= - \int \lambda_w^1 \mu_0(dw) - \sum c_i \nu_i \\ &= - \sum_{i=0}^n c_i \int \lambda_w^1 \tau_i(dw) - \sum_{i=1}^n c_i \nu_i, \end{aligned}$$

so the span of $\tau_0 - \int \lambda_w^1 \tau_0(dw)$, $\tau_i - \int \lambda_w^1 \tau_i(dw) - \nu_i$, $i = 1, \dots, n$, contains $(\operatorname{Re} A)^\perp$.

[REMARK 3.3. One further consequence of this proof should be noted. We have seen T^* has a unique fixed point τ_0 . On the other hand T^* carries the set of probability measures on X into itself, so that some probability measure is a fixed point (most simply by the Markov-Kakutani theorem [DS]). Thus we can take $\tau_0 \geq 0$, and the first of the above real orthogonal measures, $= \tau_0$ on ∂_0 and $-\int \lambda_w^1 \tau_0(dw)$ on ∂_1 , is ≥ 0 on ∂_0 and ≤ 0 on ∂_1 . This leads to some similarity between the situation for certain $R(K)$ and the familiar one when K is an annulus, as we shall note later (cf. 3.13).]

3.4. In order to apply Lemma 3.2 to $R(K)$ we make the following observations which will reduce our considerations to connected K . Let F be a component of K . Then

(a) the closure of $R(K)|F$ in $C(F)$ is $R(F)$: for each component V of $S \setminus F$ meets one of $S \setminus K$, so $R(K)|F$ contains the function $1/(z-v)$ for some $v \in V$. But in order to generate $R(F)$ it is only necessary to use a pole at one element of each component of $S \setminus F$, so $R(K)|F$ is dense in $R(F)$ as asserted. As a consequence

(b) $R(F)^\perp$ consists of the measures in $R(K)^\perp$ carried by F , and $(\operatorname{Re} R(F))^\perp$ consists of those in $(\operatorname{Re} R(K))^\perp$ carried by F .

The standard fact that we may apply a function analytic near the spectrum of an element f of $R(K)$ to f and obtain an element of $R(K)$ yields for $f(z) \equiv z$ (which has spectrum $= K$), the fact that we have a sequence $\{f_n\}$ of idempotents in $R(K)$ which tend pointwise to χ_F . This shows the first part of

(c) $z \in F$ implies M_z is carried by F , and coincides with the set of measures on ∂F representing z on $R(F)$ (i.e., $M_z(R(K)) = M_z(R(F))$).

(The second assertion follows from the first and (a)). Another consequence of the existence of $\{f_n\}$ is that no set of antisymmetry [Gl 1] meets F and its complement; since F is any component of K , and any extreme element of the unit ball of $R(K)^\perp$ is carried by a set of antisymmetry [Gl 1], we have

(d) any extreme point of the unit ball of $R(K)^\perp$ is carried by some component of K . And

(e) any extreme point μ of the unit ball of $(\operatorname{Re} R(K))^\perp$ is carried by some component F of K , and so lies in $(\operatorname{Re} R(F))^\perp$.

Indeed μ is carried by a set of antisymmetry by just the argument of [Gl 1] since for any $f \in R(K)$ real on the carrier of μ we clearly have $f\mu \in (\text{Re } R(K))^\perp$. The last assertion follows from (b).

Now suppose we knew that for every connected K for which $S \setminus K$ has finitely many components the dimension of $(\text{Re } R(K))^\perp$ is one less than the number of components of ∂K , and we are given a *disconnected* K . As is easily seen ∂F is connected for all components F of K except for a finite collection, F_1, \dots, F_k say, and for those with ∂F connected we can apply the known result (since F is connected and $S \setminus F$ has finitely many components) to conclude that $(\text{Re } R(F))^\perp = 0$. Thus the μ in (e) are each carried by one F_i , and we see that $(\text{Re } R(K))^\perp = \sum_{i=1}^k (\text{Re } R(F_i))^\perp$, and so has a (finite) dimension equal to $\sum_{i=1}^k n_i$ if ∂F_i has $n_i + 1$ components.

Now to see that (for K connected) $(\text{Re } R(K))^\perp$ has dimension n if ∂K has $n + 1$ components we need only prove $R(K)$ is dirichlet on ∂K when ∂K is connected. This reduction to the case $n = 0$ is obtained from 3.2 by induction, as follows.

Since K is connected each component V (of the finitely many) of $S \setminus K$ is simply connected and has a connected boundary $[N]$, and thus each component of ∂K is the boundary of a union of such V . Let B_0, \dots, B_n denote the components of ∂K , U_0, \dots, U_n the corresponding unions of V 's with $\partial U_i = B_i$. If any B_i does not separate the remaining components of ∂K (i.e., all others lie in one component of $S \setminus B_i$) we can apply 3.2 to $K_0 = S \setminus U_i$, $K_1 = S \setminus \bigcup_{j \neq i} U_j$ (so $K = K_0 \cap K_1$) and $R(K_0)$, $R(K_1)$ since $R(K_0)$ is dirichlet on $\partial K_0 = B_i$ and $(\text{Re } R(K_1))^\perp$ has dimension $n - 1$ by our induction hypothesis; and so $(\text{Re } R(K))^\perp$ has dimension $\leq n$ (and $\geq n$ by footnote 6).

But some B_i does not separate the others by a simple argument. If for example B_i separates so that, say, $B_{i+1}, B_{i+2}, \dots, B_n$ lie in one component, with B_0, \dots, B_{i-1} in others then B_i, B_{i+1}, \dots, B_n lie in one component of $S \setminus B_{i-1}$: for B_{i-1} lies in one (simply connected) component V of $S \setminus B_i$ which does not meet B_{i+1}, \dots, B_n , or B_i , so that the connected complement V' contains B_i, B_{i+1}, \dots, B_n and must lie in one component of $S \setminus B_{i-1}$. So another choice of the separating B_i leads to a larger number of B_j in one component, and we can continue until we arrive at a non-separating one.

In summary then

(f) *once we see $R(K)$ is dirichlet on ∂K when K and ∂K are connected (and $S \setminus K$ has finitely many components, of course) then whenever $S \setminus K$ has finitely many components if F_1, \dots, F_k are the components of K with disconnected boundaries, while ∂F_i has $n_i + 1$ components, $(\text{Re } R(K))^\perp$ has dimension $\sum_{i=1}^k n_i$.*

Our discussion also shows how Walsh's earlier theorem [Wa 1] (on sets K with $S \setminus K$ connected, cited at the beginning of this section) implies directly that $(\text{Re } R(K))^\perp$ is finite dimensional (with K as above). Indeed we need only see this when K is connected, and then components V of $S \setminus K$ are simply connected; shrinking every V slightly to a smaller simply connected \tilde{V} we can obtain a larger K , \tilde{K} , for which the components of $S \setminus \tilde{K}$ have disjoint closures, while each $R(S \setminus \tilde{V})$ is dirichlet on

$\partial \tilde{V} = \partial(S \setminus \tilde{V})$ by Walsh's first result. As in the above argument we can then use induction and Lemma 3.2 to see $(\operatorname{Re} R(\tilde{K}))^\perp$ is finite dimensional, and 3.1 then yields the same fact for K .

[(g) One final remark should be made on the special case in which K is connected and ∂K has components B_0, \dots, B_n none of which separates the remainder: then one has a basis τ_1, \dots, τ_n of $(\operatorname{Re} R(K))^\perp$ with $\tau_i \geq 0$ on B_i , ≤ 0 on $\bigcup_{j \neq i} B_j$ (and nonvanishing on each such B_j). Indeed using B_i as ∂_0 in Remark 3.3 we obtain a $\tau_i \geq 0$ on B_i , ≤ 0 elsewhere. To see the measures τ_1, \dots, τ_n we obtain for $i = 1, \dots, n$ are independent we note that since $\tau_i \perp 1$,

$$\sum_{j=1}^n \tau_i(B_j) = -\tau_i(B_0) \geq 0,$$

and that our result follows if we have strict inequality for every i : for then the matrix (a_{ij}) , $a_{ij} = \tau_i(B_j)$, $i, j = 1, \dots, n$, has $a_{ii} > \sum_{j \neq i} |a_{ij}|$ (i.e., $\tau_i(B_i) > \sum_{j \neq i, 0} -\tau_i(B_j)$), so that (a_{ij}) is nonsingular by the familiar Gerschgorin criterion [MM].

Now if $\tau_i(B_0) = 0$ we have $\lambda_z^1(B_0) = 0$ for some z in B_i since off B_i , $\tau_i = -\int_{B_i} \lambda_z^1 \tau_i(dz)$ as in 3.3 (where λ_z^1 is harmonic measure for z relative to K_1 as above, in the argument establishing (f)). But then $\lambda_z^1(B_0) = 0$ for all z in the component U of K_1^0 containing B_i . If we let U_j again be the union of the components of $S \setminus K$ with boundary $\subset B_j$ then one component W_j of $S \setminus B_j$, indeed of $S \setminus U_j^-$, contains the remaining B_k by hypothesis, and by [N, p. 101, 8.2] $W_j \cap W_k$ is the component of $(S \setminus U_j^-) \cap U_k^-$ (hence of $S \setminus (B_j \cup B_k)$) containing the remaining B_l , since $U_k^- \subset S \setminus U_j^-$, the U_i^- being pairwise disjoint. And clearly $\partial(W_j \cap W_k) = \partial W_j \cup \partial W_k$. Applying the result several times we see $\bigcap_{j \neq i} W_j$ is the component U of $K_1^0 = S \setminus (\bigcup_{j \neq i} U_j^-)$ containing B_i , and $\partial U = \bigcup_{j \neq i} \partial W_j$. Thus $\emptyset \neq \partial U \cap \partial W_0 \subset \partial U \cap B_0$. But $\lambda_z^1(B_0) = 0$ for z in U and therefore in $\partial U \cap B_0$ since $\hat{\chi}: z \rightarrow \lambda_z^1(B_0)$ is the continuous harmonic function on K_1 corresponding to the element χ_{B_0} of $C(\partial K_1)$; but for the same reason $\hat{\chi}(z) = 1$ on B_0 , our contradiction, and we conclude that $\tau_i(B_0) \neq 0$, $i \neq 0$. By the same argument $\tau_i(B_j) \neq 0$ if $i \neq j$.]

3.5. In order to apply the results of §2 to rational approximation we shall need three lemmas. The first reproduces a step in [Gl W] to show we have no completely singular measures in $R(K)^\perp$ when $S \setminus K$ has finitely many components. The result depends on the well-known result of Bishop [B 2] that $R(K) = C(K)$, if (and only if) the minimal boundary for $R(K)$ fills out all of K except for a set of plane measure zero (so in particular if ∂K is the minimal boundary for $R(K)$, hence for $R(\partial K)$, $R(\partial K) = C(\partial K)$).

LEMMA 3.6. *If $R(\partial K) = C(\partial K)$, and we have a dominant representing measure on $R(K)$ for each $z \in K$, then there are no completely singular measures orthogonal to $R(K)$.*

Further, suppose we also have ∂K the minimal boundary for $R(K)$ (so that $R(\partial K)$

$= C(\partial K)$ and each nontrivial Gleason part for $R(K)$ is a union of components⁽⁸⁾ of K^0 . If Q is any collection of nontrivial parts, with (open) union U , and we select one dominant representing measure λ for one point in each part not in Q to form the collection Λ , and set $K_Q = K \setminus U$ then

$$(3.7) \quad R(K_Q)^\perp = \sum_{\Lambda} R(K)_\lambda^\perp$$

(where the sum denotes all norm convergent series as before). Finally, if in addition harmonic measures are dominant, $R(K_Q)^\perp = R(K_Q^0)^\perp$.

Note that $\partial K_Q = \partial K$, so that in the second part of 3.6 ∂K_Q is the minimal boundary of the algebra $R(K_Q) \supset R(K)$. (We can naturally view $R(K)$ as a subalgebra of $R(K_Q)$ by identifying $R(K)$ and $R(K)|_{K_Q}$.) To begin the proof note that if σ is a completely singular element of $R(K)^\perp$, and λ is any dominant representing measure for $z_0 \in K^0$, then

$$\sigma/(z - z_0) - \sigma(1/(z - z_0)) \cdot \lambda$$

is orthogonal to $C + (z - z_0) \cdot R(K) = R(K)$ so by Theorem 2.6 its singular part with respect to λ , $\sigma/(z - z_0)$, is also. This same argument applies to $\sigma^\perp = \sigma/(z - z_0)$ to show

$$1/(z - z_0) \cdot 1/(z - z_1) \cdot \sigma$$

is orthogonal to $R(K)$, $z_1 \in K^0$, and continuing we see σ is orthogonal to the subalgebra of $C(\partial K)$ generated by $R(K)$ and all functions $z \rightarrow 1/(z - z_0)$, $z_0 \in K^0$, hence to $R(\partial K) = C(\partial K)$. So $\sigma = 0$.

Similarly if we now take $z_0 \in U$ with dominant representing measure λ_0 , then $\lambda \in \Lambda$ and λ_0 are mutually singular and if $\mu \in R(K)_\lambda^\perp$, we have

$$\mu/(z - z_0) - \mu(1/(z - z_0))\lambda_0$$

orthogonal to $C + (z - z_0)R(K) = R(K)$, so that $\mu/(z - z_0) \perp R(K)$; as before we can conclude that $\mu \perp R(K_Q)$.

Now since we may view $R(K)$ as a subalgebra of $R(K_Q)$ in $C(\partial K)$, each measure multiplicative on $R(K_Q)$ is multiplicative on $R(K)$, and of course represents a point $z \in K_Q$. On the other hand for $z \in K_Q^0$ if $\lambda^1 \in M_z$ for $R(K)$, and so is in $L^1(\lambda)$ for some λ in Λ , then $\mu = \lambda_z - \lambda^1$ is orthogonal to $R(K)$, and thus orthogonal to $R(K_Q)$ as we have just seen. Since λ_z is multiplicative on $R(K_Q)$, the same is true of λ^1 : so $z \in K_Q^0$ has the same set of representing measures on $\partial K_Q = \partial K$ for $R(K)$ and $R(K_Q)$. The same is of course the case for $z \in \partial K_Q$, a peak point, so that the representing measures for $R(K_Q)$ are just those for $R(K)$ which represent points in K_Q .

We have two consequences: dominant representing measures are available for $R(K_Q)$ (so that $R(K_Q)^\perp$ has no completely singular elements by the argument

⁽⁸⁾ Each component lies in a part P , and since boundary points form singleton parts, $P \subset K^0$, and so is a union of components of K^0 .

applied to $R(K)$) and the nontrivial parts for $R(K_Q)$ are simply the nontrivial parts for $R(K)$ contained in K_Q , i.e., those not in Q . Thus $R(K_Q)^\perp = \sum_\Lambda R(K_Q)_\lambda^\perp$ by 2.8.

But since $R(K) \subset R(K_Q)$ we have $R(K_Q)^\perp \subset R(K)^\perp$, and so $R(K_Q)_\lambda^\perp \subset R(K)_\lambda^\perp$; on the other hand we have seen $R(K)_\lambda^\perp \subset R(K_Q)^\perp$, hence $\subset R(K_Q)_\lambda^\perp$, for $\lambda \in \Lambda$, so $R(K_Q)_\lambda^\perp = R(K)_\lambda^\perp$, and this implies (3.7).

For the final fact, that $R(K_Q)^\perp = R(K_Q^0)^\perp$ when harmonic measures are dominant, note first that then each measure in the first set is carried by ∂K_Q^0 since each λ in Λ is. Since each component of $S \setminus K_Q^0$ must meet a component of $S \setminus K_Q$ (∂K_Q being nowhere dense) and since only one pole is needed in each complementary component to generate $R(K_Q^0)$, $R(K_Q)|_{K_Q^0}$ is dense in $R(K_Q^0)$, so that $R(K_Q)^\perp \subset R(K_Q^0)^\perp$. The reverse inclusion is trivial, and our proof complete.

The crucial property of $R(K)$, that $R(K) = C + (z - z_0)R(K)$ for $z_0 \in K^0$, is shared by $A(K)$, so by the same arguments we obtain

COROLLARY 3.7. *If K satisfies the hypotheses of 3.6, while U , Q and Λ are defined in terms of the parts for $A(K)$, then $\sum_\Lambda A(K)_\lambda^\perp = B^\perp$, where B is the subalgebra of $C(K_Q)$ generated by $A(K)$ and all functions $z \rightarrow 1/(z - z_0)$, $z_0 \in U$.*

Note that since fewer measures on ∂K represent $z \in K$ on $A(K)$ than on $R(K)$, we have dominant representing measures for $A(K)$ by hypothesis.

Lemma 3.6 will (later) provide some easy extensions of Mergelyan's results, but now we should note

COROLLARY 3.8. *Under the hypotheses of 3.6 (respectively 3.7) $\sum_\Lambda R(K)_\lambda^\perp$ (respectively $\sum_\Lambda A(K)_\lambda^\perp$) is weak* closed.*

In particular each summand $R(K)_\lambda^\perp$ is a weak* closed space of measures.

Finally the argument in 3.6 showing there are no completely singular orthogonal measures can be used in other instances when the crucial property that $A = C + (z - z_0)A$ holds for sufficiently many z_0 : for example, the algebra A mentioned after the statement of 3.2, consisting of all functions in $C(D)$ analytic on $U = D^0 \setminus \bigcup_{i=0}^n \partial_i$ has this property for all z_0 in U ; since each point z of ∂U is a peak point for $R(\partial U)$ — z in ∂_i being a peak point for Wermer's algebra A_i —the cited theorem of Bishop [B 2] applies to show $R(\partial U) = C(\partial U)$, and thus A^\perp contains no completely singular measures. The same applies to its subalgebra B generated by all functions analytic on $D \setminus \bigcup_{i=0}^n \partial_i$, i.e., each analytic on $(1 + \varepsilon)D \setminus \bigcup_{i=0}^n \partial_i$ for some $\varepsilon > 0$. Both algebras have $\mathfrak{M} = D$, as one sees from [Ar], and since $(\operatorname{Re} B)^\perp$ and $(\operatorname{Re} A)^\perp$ both have dimension $n + 1$ (by the argument given after the statement of 3.2) $A = B$ by 2.11.

Our second lemma is an assertion about harmonic measures which must be well known, although we have not been able to find it in the literature; in the simplest case it essentially says that if we punch a hole in a domain V to obtain a new one U the old harmonic measure (for $z \in U$) is absolutely continuous with respect to the new one away from the boundary of the hole. I am indebted to Maynard G. Arsove for suggesting its proof.

We shall use the following notation. Since K^0 has countably many components, we select one point z_n in the n th component, and let λ_n be its harmonic measure. Then λ^* will denote the measure $\sum 2^{-n}\lambda_n$.

LEMMA 3.9. Let $K=K_0 \cap K_1$ where each compact K_i has its boundary as the minimal boundary for $R(K_i)$ and $\partial K = \partial K_0 \cup \partial K_1$. Let λ_z^i be the harmonic measure for $z \in K_i$ relative to K_i , and λ_z that for $z \in K$ relative to K .

Let U be a component of K^0 , and U_i the component of K_i^0 containing U (so $U=U_0 \cap U_1$). Then for $z \in U$, λ_z^0 (respectively λ_z^1) is absolutely continuous with respect to λ^* on $\partial U_0 \setminus \partial K_1$ (respectively $\partial U_1 \setminus \partial K_0$).

In particular, if $(\partial U_0 \setminus \partial K_1) \cup (\partial U_1 \setminus \partial K_0) \subset \partial U$, $\lambda_z^0 + \lambda_z^1$ is absolutely continuous with respect to λ_z on $\partial U \setminus (\partial K_0 \cap \partial K_1)$ ($\subset (\partial U_0 \setminus \partial K_1) \cup (\partial U_1 \setminus \partial K_0)$).

Proof. For the first part we need only show that for any compact $F \subset \partial U_0 \setminus \partial K_1$ we have $\lambda_z^0 F = 0$ if $\lambda^* F = 0$. Since the harmonic measures for points in a given component of K^0 are equivalent, $\lambda^* F = 0$ and our definition of λ^* imply $\lambda_z F = 0$ for every z in K^0 .

Since F is a G_δ , we can select a sequence $\{h_n\}$ in $C(\partial K) = C(\partial K_0 \cup \partial K_1)$, all vanishing on a fixed neighborhood of ∂K_1 , with $h_n \searrow \chi_F$ pointwise.

Let \hat{h}_n^0 be the continuous harmonic function⁽⁹⁾ on K_0 which extends $h_n|_{\partial K_0}$; \hat{h} will denote the continuous harmonic function on K which extends $h \in C(\partial K)$. Noting that $\partial K_1 \setminus \partial K_0 \subset K \setminus \partial K_0 \subset K_0 \setminus \partial K_0 = K_0^0$, we can set

$$\begin{aligned} k_n &= 0 \text{ on } \partial K_0, \\ &= \hat{h}_n^0 \text{ on } \partial K_1 \setminus \partial K_0, \end{aligned}$$

to obtain a well-defined function on $\partial K = \partial K_0 \cup \partial K_1$ which is continuous since $\hat{h}_n^0 \rightarrow 0$ as we approach a point in $\partial K_0 \cap \partial K_1$. Similarly

$$\begin{aligned} (3.8) \quad f &= 0 \text{ on } \partial K_0, \\ &= \lambda_z^0 F \text{ on } \partial K_1 \setminus \partial K_0 \end{aligned}$$

defines an element of $C(\partial K)$ since $h_n \geq \chi_F \geq 0$ implies $\lambda_z^0(h_n) = \hat{h}_n^0(z) \geq \lambda_z^0 F \geq 0$, $z \in K_0$, so that $f \rightarrow 0$ as we approach a point of $\partial K_0 \cap \partial K_1$.

Now

$$z \rightarrow \hat{h}_n^0(z) - \hat{k}_n(z)$$

is a continuous harmonic function on K with a restriction to ∂K which is h_n on ∂K_0 and $\hat{h}_n^0 - \hat{k}_n = 0 = h_n$ on $\partial K_1 \setminus \partial K_0$, hence is h_n on all of $\partial K = \partial K_0 \cup \partial K_1$. So

$$(3.9) \quad \hat{h}_n^0(z) - \hat{k}_n(z) = \hat{h}_n(z), \quad z \in K.$$

As $n \rightarrow \infty$, $h_n \searrow \chi_F$ pointwise on ∂K , and so $\hat{h}_n(z) = \lambda_z(h_n) \searrow \lambda_z F$, $z \in K$, and

⁽⁹⁾ I.e., continuous on K_0 and harmonic on K_0^0 . Such an extension exists since ∂K is the minimal boundary for $R(K)$; see the introductory remarks to this section.

$\hat{h}_n^0(z) = \lambda_z^0(h_n) \searrow \lambda_z^0 F$, $z \in K_0$ by monotone convergence. The latter implies $k_n \searrow f$ (defined in (3.8)), so that $\hat{k}_n \searrow \hat{f}$, again by monotone convergence. By (3.9) then

$$(3.10) \quad \lambda_z^0 F - \hat{f}(z) = \lambda_z F = 0, \quad z \in K^0,$$

so $\lambda_z^0 F = \hat{f}(z)$ on K^0 . Since \hat{f} is continuous on K , and vanishes on ∂K_0 by (3.8), we conclude that

$$\lim_{z \rightarrow z_0; z \in K^0} \lambda_z^0 F = 0, \quad z_0 \in (\partial K_0) \cap K^{0-},$$

so that for z_0 in a neighborhood V of F in $\partial U_0 \setminus \partial K_1$ ($\subset (\partial K_0) \cap K^{0-}$)

$$\lim_{z \rightarrow z_0; z \in K_0^0} \lambda_z^0 F = 0.$$

(Note that $K_0 \cap W = K \cap W$ for a small neighborhood W of z_0 .)

On the other hand this also holds for z_0 in $\partial U_0 \setminus V$ since the harmonic extension to K_0 of any $g \geq \chi_F$ in $C(\partial K_0)$ vanishing off V tends to zero as we approach z_0 . So the harmonic function $z \rightarrow \lambda_z^0 F$ on U_0 tends to zero as we approach ∂U_0 , and thus must vanish on all of U_0 , in particular at $z \in U \subset U_0$, yielding our first assertion.

In the particular case that $(\partial U_0 \setminus \partial K_1) \cup (\partial U_1 \setminus \partial K_0) \subset \partial U$ the proof (starting now with $F \subset \partial U_0 \setminus \partial K_1$ having $\lambda_z F = 0$) is unchanged up to (3.10), the latter part of which now only holds for z in U . But replacing K^0 by U the argument now leads to the same conclusion that $\lambda_z^0 F \rightarrow 0$ as $z \rightarrow z_0$ from within U_0 (for z_0 in a neighborhood of F in $\partial U_0 \setminus \partial K_1$) since $\partial U_0 \setminus \partial K_1 \subset U^- \cap \partial K_0$ while $U \cap W = U_0 \cap W$ for a small neighborhood W of z_0 . The remainder of the argument is unchanged.

LEMMA 3.10. *Suppose the harmonic measure $\lambda = \lambda_{z_0}$ is a dominant representing measure on $R(K)$, while $(\operatorname{Re} R(K))_\lambda^\perp$ is at most one-dimensional and the component U of K^0 containing z_0 is a part for $R(K)$ with ∂U in the minimal boundary. Then*

(a) $\lambda = \lambda_{z_0}$ is strongly dominant for $R(K)$ (hence for $A(K)$ as well), and $R(K)_\lambda^\perp = A(K)_\lambda^\perp$.

(b) If U is nicely connected (§1), λ is the unique representing measure for z_0 on $R(K)$.

Note that if the conclusion of (a) holds for some z_0 in each Gleason part for $R(K)$, while ∂K is the minimal boundary for $R(K)$, then 3.6 and 2.8 (or (2.3)) combine to show $R(K) = A(K)$.

To begin our proof of 3.10, by (2.3c) we have a strongly dominant representing measure for z_0 , necessarily of the form $g\lambda$ for $g \in L^1(\lambda)$ since λ is dominant. Since $Cg\lambda \geq \lambda$ for some constant C , g is bounded below; to establish the first assertion of (a) we need only show g is bounded. Of course we shall assume $g \not\equiv 1$, and since $\int g d\lambda = 1$, $g > 1$ and $g < 1$ on sets of positive λ measure, and $g - 1$ cannot be of one sign.

Now $(g - 1)\lambda$ is a real nonzero measure orthogonal to $R(K)$, hence spans $(\operatorname{Re} R(K))_\lambda^\perp$. So by Corollary 2.10 we have

$$(3.11) \quad R(K)_\lambda^\perp = R(K)_{g\lambda}^\perp = C(g - 1)\lambda + H_0^1(g\lambda) \cdot g\lambda.$$

We have $(g-1)r\lambda \in R(K)^\perp$ for $r \in R(K)$, and for $r(z) \equiv z$ by (3.11)

$$z(g-1) = z_1(g-1) + fg, \quad z_1 \in C, \quad f \in H_0^1(g\lambda),$$

so $(z-z_1)(1-1/g) = f \in H_0^1(g\lambda)$. So there are r_n in $R(K)$, $r_n \rightarrow (z-z_1)(1-1/g)$ in $L^1(g\lambda)$ (hence also in $L^1(\lambda)$), and for $r \in R(K)$

$$\int \left(1 - \frac{1}{g}\right)(z-z_1)r \, d\lambda = \lim \int r_n r \, d\lambda = \lim \int r_n r g \, d\lambda = \int (g-1)(z-z_1)r \, d\lambda = 0.$$

Thus $(1-1/g)\lambda$ is orthogonal to $(z-z_1) \cdot R(K)$.

Of course if $z_1 \notin K$ this implies the real measure $(1-1/g)\lambda$ is orthogonal to $R(K)$, whence $(1-1/g)$ is a multiple of $g-1$: $1-1/g = (g-1)/g = c(g-1)$. Since this implies $g=1$ or $g=1/c$ a.e. λ (so $g-1$ cannot change sign), we conclude that $z_1 \in K$.

Now $(1-1/g)\lambda$ is orthogonal to $(z-z_1)R(K)$, hence to any rational function in $R(K)$ vanishing at z_1 . But any r in $R(K)_{z_1} = \{r \in R(K) : r(z_1) = 0\}$ is the uniform limit of such rational functions, so that $(1-1/g)\lambda$ is orthogonal to the maximal ideal $R(K)_{z_1}$. For the real number $k = -\lambda(1-1/g)$ we thus have

$$(3.12) \quad (1-1/g)\lambda + k\lambda_{z_1}$$

orthogonal to $C + R(K)_{z_1} = R(K)$.

If z_1 does not lie in the component U of K^0 containing z_0 , hence is not in the Gleason part containing z_0 , then λ_{z_1} and $\lambda = \lambda_{z_0}$ are mutually singular (else $\|\lambda_{z_1} - \lambda\| < 2$), and so by 2.6 $(1-1/g)\lambda \in R(K)^\perp$ so $(1-1/g) = (g-1)/g = c(g-1)$, whence $g \equiv 1$ as earlier. And of course the same occurs if $k=0$, so we must have $k \neq 0$.

Thus $z_1 \in U$, and the Radon-Nykodym derivative $h = d\lambda_{z_1}/d\lambda$ is bounded (as is $h^{-1} = d\lambda/d\lambda_{z_1}$) and satisfies

$$(3.13) \quad (1-1/g) + kh = c(g-1)$$

for some real c , since (3.12) is now an element of $(\text{Re } R(K))_\lambda^\perp$. This shows g is bounded since the left side of (3.13) is bounded while $c=0$ implies $(g-1)/g = -kh$ is of one sign, which is impossible if $g \neq 1$. So we have the first assertion of (a).

For the second assertion, that $R(K)_\lambda^\perp = A(K)_\lambda^\perp$, we need only show $H^2(R(K), \lambda) = H^2(A(K), \lambda)$ by Corollary 2.12, since λ is strongly dominant. But

$$(3.14) \quad H^2(A(K), \lambda) \ominus H^2(R(K), \lambda) = H_0^2(A(K), \lambda) \ominus H_0^2(R(K), \lambda)$$

is orthogonal in $L^2(\lambda)$ to $\overline{H^2(A(K), \lambda)}$, so to any element of $R(K) + \overline{R(K)}$, hence to $\text{Re } R(K)$. The same of course applies to any element of

$$(3.15) \quad \overline{H_0^2(A(K), \lambda)} \ominus \overline{H_0^2(R(K), \lambda)}$$

which is the space of conjugates to elements of (3.14), and which is also *orthogonal* to (3.14). So if (3.14) is nonzero we have $(\text{Re } R(K))_\lambda^\perp$ a (complex) space of dimension at least two, showing (3.14) = $\{0\}$ and completing our proof of (a).

We obtain (b) by transporting matters to the unit disc.

Since U is nicely connected we have a Riemann map ρ of the unit disc

$$\{w \in \mathbb{C} : |w| < 1\}$$

onto U with $\rho(0) = z_0$, and a 1-1 map σ defined on a subset of full measure of the unit circle which satisfies $\sigma(e^{i\theta}) = \lim_{r \rightarrow 1} \rho(re^{i\theta})$, with values in ∂U of course. For a continuous harmonic function f on U^- we have

$$(3.16) \quad \int f d\lambda = f(\rho(0)) = \int f(\rho(re^{i\theta})) \frac{d\theta}{2\pi} = \lim_{r \rightarrow 1} \int f(\rho(re^{i\theta})) \frac{d\theta}{2\pi} = \int f(\sigma(e^{i\theta})) \frac{d\theta}{2\pi}$$

by dominated convergence, so

$$(3.17) \quad \int f d\lambda = \int f \circ \sigma \frac{d\theta}{2\pi}$$

for any f in $C(\partial U)$, hence for any bounded Baire function. Thus $f \rightarrow f \circ \sigma$ maps $L^p(\lambda)$ isometrically onto $L^p(\mu)$, where $d\mu = d\theta/2\pi$, while $g \rightarrow g \circ \sigma^{-1}$ provides the inverse, for all p .

Now let H^p temporarily represent (the boundary value version of) the usual Hardy space on the unit circle. Since $f \circ \rho \in R(K) \circ \rho$ is a bounded analytic function, and $f \circ \sigma(e^{i\theta}) = \lim_{r \rightarrow 1} f(\rho(re^{i\theta}))$ on the domain of σ , $R(K) \circ \sigma \subset H^\infty$, and

$$(3.18) \quad R(K) \subset H^\infty \circ \sigma^{-1}.$$

Taking $L^2(\lambda)$ closures we obtain

$$(3.19) \quad H^2(R(K), \lambda) \subset H^2 \circ \sigma^{-1}.$$

Now λ is of course multiplicative on $H^2 \circ \sigma^{-1}$ (by (3.17)) so, as earlier,

$$(3.20) \quad (H^2 \circ \sigma^{-1}) \ominus H^2(R(K), \lambda) = (H^2 \circ \sigma^{-1})_0 \ominus H_0^2(R(K), \lambda)$$

(where the noughts denote elements of mean zero) is orthogonal in $L^2(\lambda)$ to $\bar{H}^2 \circ \sigma^{-1}$, hence to any element of $R(K) + \bar{R}(K)$, or to $\text{Re } R(K)$. The same of course applies to

$$(3.21) \quad \overline{(H^2 \circ \sigma^{-1})} \ominus \overline{H^2(R(K), \lambda)},$$

the space of conjugates to elements of (3.20) and also *orthogonal* to (3.20). So (3.20) is nonzero only if we have $(\text{Re } R(K))_\lambda^\perp$ at least two dimensional.

Thus we conclude that

$$(3.22) \quad H^2(R(K), \lambda) = H^2 \circ \sigma^{-1}.$$

Since $g-1$ is bounded, and orthogonal to $R(K)$ in $L^2(\lambda)$, it is also orthogonal to (3.22), and for all f in H^∞

$$0 = \int (f \circ \sigma^{-1})(g-1) d\lambda = \int f \cdot (g-1) \circ \sigma \frac{d\theta}{2\pi}$$

by (3.17). Consequently $(g-1) \circ \sigma = 0$ a.e. on the circle, whence $g-1=0$ a.e. λ , completing our proof.

We can now proceed to our main applications to rational approximation. The first will be the cited result of Ahern and Sarason [AS 2] giving the rational defect when $S \setminus K$ has finitely many components. In [AS 2] this result was obtained from consequences of Ahern and Sarason's earlier detailed investigations of hypo-dirichlet algebras [AS 1]; aside from what we have developed so far our proof will use well-known facts valid when representing measures are unique, along with two relatively elementary and accessible facts given by Ahern and Sarason [AS 2, Lemmas 1-3]: *if $S \setminus K$ has finitely many components, then points in distinct components of K^0 have mutually singular harmonic measures⁽¹⁰⁾; moreover the simply connected components of K^0 are nicely connected.*

We begin with the following special case of the Ahern-Sarason result, from which the general result will follow via 3.2.

PROPOSITION 3.11 ([AS 2]). *Suppose $S \setminus K$ has $n+1$ components H_0, H_1, \dots, H_n , while each component of K has a connected boundary (or, alternatively, each component of K^0 is simply connected). Then $R(K)$ is dirichlet on ∂K , and $A(K) = R(K)$.*

Of course the second assertion follows from the first by 2.13. We shall proceed by induction on n , assuming the result for $n-1$ (the case $n=0$ is of course Walsh's result [Wa 1]); by 3.4(f) we need only prove the first assertion assuming K is connected.

Since K is connected each "hole" H_i is simply connected; and so has a connected boundary [N]. Since $\partial K = \bigcup \partial H_j$ is connected by hypotheses, we have $\bigcup H_i^-$ connected, and a simple argument⁽¹¹⁾ shows we can find an i for which $\bigcup_{j \neq i} H_j^-$ is connected; we shall take $i=0$, and so assume $\bigcup_{j \neq 0} H_j^-$, and thus $\bigcup_{j \neq 0} \partial H_j$, is connected.

Now set $K_0 = S \setminus H_0$, $K_1 = S \setminus \bigcup_{j \neq 0} H_j$, so that ∂K_i is connected, $K = K_0 \cap K_1$, $\partial K = \partial K_0 \cup \partial K_1$, and, by our induction hypothesis, $R(K_i)$ is dirichlet on ∂K_i . Moreover H_0 is the union of an increasing sequence of simply connected domains H_0^ν , with $H_0^\nu \subset H_0$, so that K is the intersection of a decreasing sequence $K^\nu = K_1 \cap K_0^\nu$, $K_0^\nu = S \setminus H_0^\nu$; since $\partial K_0^\nu = \partial H_0^\nu$ is connected, $R(K_0^\nu)$ is dirichlet on ∂K_0^ν so the rational defect for K^ν is ≤ 1 by 3.2, and therefore the same is true for K by sweeping, as in 3.1.

By 3.6 we have no completely singular measures orthogonal to $R(K)$, and if each representing measure $\lambda = \lambda_z$ is unique then by [H1, 6.7, and L] there are no nonzero real measures in $R(K)_\lambda^\perp$, hence none in $R(K)^\perp$ by Theorem 2.8. So we can assume we have a z in K with a representing measure $\mu \neq \lambda_z$ carried by ∂K . Since the rational defect is at most 1, $\sigma = \mu - \lambda_z$ must span $(\text{Re } R(K))^\perp$.

⁽¹⁰⁾ Note that two such components can share the same boundary; cf. [K].

⁽¹¹⁾ Very similar to that used before 3.4(f).

For our compact K_i (above), let $\partial_0 = \partial K_0 = \partial H_0$, $\partial_1 = \partial K_1 = \bigcup_{j \neq 0} \partial H_j$. Since $R(K_i)$ is dirichlet on ∂_i , by sweeping we obtain as earlier⁽¹²⁾ (cf. 3.2)

$$(3.23) \quad \begin{aligned} \sigma_{\partial_0 \setminus \partial_1} + \int \lambda_w^0 \sigma_{\partial_1}(dw) &= 0 \\ \sigma_{\partial_1} + \int \lambda_w^1 \sigma_{\partial_0 \setminus \partial_1}(dw) &= 0, \end{aligned}$$

so that $\sigma_{\partial_0 \setminus \partial_1} \neq 0$ (else $\sigma=0$) and

$$\sigma_{\partial_0 \setminus \partial_1} = \int \eta_t \sigma_{\partial_0 \setminus \partial_1}(dt),$$

where $\eta_t = \int \lambda_w^0 \lambda_t^1(dw)$. So

$$\sigma_{\partial_0 \setminus \partial_1} = \int (\eta_t)_{\partial_0 \setminus \partial_1} \sigma_{\partial_0 \setminus \partial_1}(dt),$$

and if p is a unimodular function with $p\sigma_{\partial_0 \setminus \partial_1} = |\sigma_{\partial_0 \setminus \partial_1}|$ we have

$$\|\sigma_{\partial_0 \setminus \partial_1}\| = \sigma_{\partial_0 \setminus \partial_1}(p) = \int (\eta_t)_{\partial_0 \setminus \partial_1}(p) \sigma_{\partial_0 \setminus \partial_1}(dt)$$

so that, since $\|\eta_t\|=1$, $(\eta_t)_{\partial_0 \setminus \partial_1}(p)=1$ and $\eta_t(\partial_0 \setminus \partial_1)=1$ for some t in $\partial_0 \setminus \partial_1$.

Now H_0 lies in one component W of K_1^0 , so that $\partial_0 \setminus \partial_1 \subset H_0^- \setminus \partial_1 \subset W^- \setminus \partial W = W$. Thus the second half of (3.23) implies ∂W carries σ_{∂_1} since each λ_w^1 with $w \in \partial_0 \setminus \partial_1$ is carried by ∂W . Moreover our t with $\eta_t(\partial_0 \cap \partial_1)=0$ lies in W , so

$$(3.24) \quad 0 = \int \lambda_w^0(\partial_0 \cap \partial_1) \lambda_t^1(dw).$$

But the closed support of λ_t^1 is precisely all of ∂W (otherwise W and some H_j , $j \neq 0$, lie in the same component V of the complement of the support, and the analytic function $z \rightarrow \int ((w-t)/(w-z)) \lambda_t^1(dw)$ on V must vanish identically since it does on H_j ; and of course it cannot vanish at $z=t$). So for each component U of K_0^0 the relatively open subset $U \cap \partial W$ of the support of λ_t^1 contains a w for which $\lambda_w^0(\partial_0 \cap \partial_1)=0$ by (3.24), or is empty; since all λ_w^0 for w in U are equivalent, $\lambda_w^0(\partial_0 \cap \partial_1)=0$ for all w in $U \cap \partial W$, hence for all w in $\partial W \setminus \partial_0$ since $\partial W \setminus \partial_0 \subset K \setminus \partial_0 \subset K_0 \setminus \partial_0 = K_0 \setminus \partial K_0 = K_0^0$. As we saw $\partial W \setminus \partial_0$ carries $\sigma_{\partial_1 \setminus \partial_0}$, and since

$$\sigma_{\partial_0} + \int \lambda_w^0 \sigma_{\partial_1 \setminus \partial_0}(dw) = 0$$

we conclude that for any Borel set $F \subset \partial_0 \cap \partial_1$

$$\sigma F = \sigma_{\partial_0}(F) = - \int \lambda_w^0(F) \sigma_{\partial_1 \setminus \partial_0}(dw) = - \int 0 \cdot \sigma_{\partial_1 \setminus \partial_0}(dw) = 0.$$

So on $\partial_0 \cap \partial_1$, $\sigma=0$, or $\mu=\lambda_z$.

⁽¹²⁾ λ_w^0 is harmonic measure relative to K_0 ; thus the first measure in (3.23) is a real measure on ∂_0 orthogonal to $R(K_0)$, etc.

Now let U be the component of K^0 containing our z represented by μ and λ_z , and let U_i be the component of K_i^0 containing U , $i=0, 1$. Sweeping μ to $\partial_i = \partial K_i$ yields a measure representing z on $R(K_i)$ (hence $=\lambda_z^i$) and necessarily $\geq \mu_{\partial_i}$; thus $\mu \leq \lambda_z^0 + \lambda_z^1$. As a consequence, by Lemma 3.9, μ is absolutely continuous with respect to the measure λ^* on $\partial K \setminus (\partial_0 \cap \partial_1) = (\partial_0 \setminus \partial_1) \cup (\partial_1 \setminus \partial_0)$, and since $\mu = \lambda_z$ on $\partial_0 \cap \partial_1$ as well, $\mu \ll \lambda^*$.

Now if U is a Gleason part for $R(K)$ we have a contradiction to $\mu \neq \lambda_z$; for then μ is singular with respect to λ_w for w in any component of K^0 other than U , so $\mu \ll \lambda^*$ implies $\mu \ll \lambda_z$, and by 3.10(b), $\mu = \lambda_z$ since U is nicely connected.

So U is not a Gleason part. If $\partial U \cap (\partial H_j \setminus \bigcup_{i \neq j} \partial H_i) = \emptyset$ for some j we have a proper subset F of $\{H_0, H_1, \dots, H_n\}$ with $\partial U \subset \partial^* = \bigcup_{H \in F} \partial H$ and $\partial H \cap \partial U \neq \emptyset$ for H in F , and if we set $K^* = S \setminus \bigcup_{H \in F} H$ then clearly U cannot be an entire Gleason part for the subalgebra $R(K^*)$ of $R(K)$. On the other hand ∂^* is connected since the simply connected sets U, H_j all have connected boundaries, so $R(K^*)$ is dirichlet on ∂^* by our induction hypotheses. But the nondegenerate Gleason parts for dirichlet algebras are connected by Wermer's theorem [We 2], and so are precisely the components of K^{*0} ; U is such a component, hence is a Gleason part for $R(K^*)$ and so for $R(K)$. We conclude then that $\partial U \cap (\partial H_j \setminus \bigcup_{i \neq j} \partial H_i) \neq \emptyset$ for every j .

This implies $U \cup (\bigcup_{j \neq 0} H_j)$ lies in one component of $S \setminus H_0^- = K_0^0$ and the component is of course U_0 . As a consequence if $x \in \partial U_1 \setminus \partial K_0 = \partial U_1 \setminus \partial H_0 \subset \bigcup_{j \neq 0} \partial H_j \setminus \partial H_0$ then $x \in U_0^- \setminus \partial H_0 \subset U_0$, so clearly $x \in \partial U$.

On the other hand $\partial U_0 \setminus \partial K_1 \subset \partial U$ as well. For since H_0 lies in one component of K_1^0 , either $H_0 \subset U_1$, or $U_1 \cap H_0^- = U_1 \cap H_0 = \emptyset$ so $U_1 \subset S \setminus H_0^- = K_0^0$, whence $U_1 \subset U_0$ and $U = U_1$. In the latter case U is a component of K_1^0 , hence a part for $R(K_1)$ which is again a contradiction. So $H_0 \subset U_1$, and thus $\partial U_0 \setminus \partial K_1 \subset \partial H_0 \setminus \partial K_1 \subset U_1^- \setminus \partial K_1 \subset U_1$. So clearly $\partial U_0 \setminus \partial K_1 \subset \partial U$.

But now the last part of 3.9 says $\mu \leq \lambda_z^0 + \lambda_z^1 \ll \lambda_z$ on $\partial U \setminus (\partial K_0 \cap \partial K_1) = \partial U \setminus (\partial_0 \cap \partial_1)$ and since $\mu = \lambda_z$ on $\partial_0 \cap \partial_1$, while $\partial U_0 \cup \partial U_1 \subset \partial U \cup (\partial_0 \cap \partial_1)$ supports μ , we have $\mu \ll \lambda_z$.

Since U is not a Gleason part some w in $K^0 \setminus U$ lies in the same part. Since we now know λ_z is a dominant representing measure for $R(K)$, by 2.7 $\lambda_w \ll \lambda_z$. This of course contradicts the result of Ahern and Sarason cited immediately preceding 3.11, and so completes our proof of 3.11.

Omitting the first three and last five paragraphs of the preceding proof (and replacing H_0 by $S \setminus K_0$ and $\bigcup_{j \neq 0} H_j$ by $S \setminus K_1$) yields a proof of the following.

PROPOSITION 3.12. *Suppose K, K_0, K_1 are compact and connected, with connected boundaries, $K = K_0 \cap K_1$, $\partial K = \partial K_0 \cup \partial K_1$, and $S \setminus K_0$ lies in one component of K_1^0 . Finally, suppose the components of K^0 are nicely connected and parts for $R(K)$, $R(K_i)$ is dirichlet on ∂K_i , $i=0, 1$, and $(\text{Re } R(K))^\perp$ is at most one-dimensional.*

Then $R(K)$ is dirichlet on ∂K .

Via Lemma 3.2 we can now pass easily from 3.11 to the full Ahern-Sarason result, of which the following is a slight variant.

THEOREM 3.13. *Suppose $S \setminus K$ has finitely many components. Then $R(K) = A(K)$ and the nontrivial parts are precisely the components of K^0 , while harmonic measures are strongly dominant. If λ is the harmonic measure of a point in a component U of K^0 then $(\operatorname{Re} R(K))_\lambda^\perp$ is n -dimensional if U is n -connected (i.e., has a boundary with $n+1$ components); in fact if F_0, F_1, \dots, F_n are the components of the boundary of U , $(\operatorname{Re} R(K))_\lambda^\perp$ is spanned by measures $f_1\lambda, \dots, f_n\lambda$ where f_i is a bounded function ≥ 0 on F_i , and ≤ 0 elsewhere, for $i=1, \dots, n$.*

Because of Remarks 3.4 we need only prove 3.13 when K is connected, and then 3.4(f) and 3.11 combine to show the rational defect is at most one less than the number of components of ∂K ; since the analytic defect is at least this large (cf. footnote 6), $(\operatorname{Re} R(K))^\perp$ and $(\operatorname{Re} A(K))^\perp$ have the same dimension, so $R(K) = A(K)$ by 2.11.

Now choose a maximal set F of components of $S \setminus K$ with the property that $\bigcup_{H \in F} H^-$ is connected and set $C = \bigcup_{H \in F} H$. We shall prove 3.13 inductively assuming it holds for the compact set $K_1 = K \cup C$.

Let $K_0 = S \setminus C$, so ∂K_0 is connected and disjoint from ∂K_1 , hence lies in one component U_1 of K_1^0 , and $\partial K = \partial K_0 \cup \partial K_1$. Evidently ∂U_1 separates ∂K_0 from $K_1^0 \setminus U_1$, and thus $K_1^0 \setminus U_1$ lies in one component U_0 of K_0^0 . Now the components of K^0 are those of K_1^0 other than U_1 , $U_0 \cap U_1$, and those of K_0^0 other than U_0 . So given two components of K^0 we see they lie in distinct Gleason parts by noting that they do for one of the subalgebras $R(K_i)$ of $R(K)$.

Thus the components of K^0 are the nontrivial parts, once we see each $z \in \partial K = \partial K_0 \cup \partial K_1$ is a peak point for $R(K)$; but z is a peak point for $R(K_i) \subset R(K)$, $i=0$ or 1 since we are assuming 3.13 for K_1 (and it holds for K_0 by 3.11).

For the same reason we can easily see λ_z ($z \in U$) is dominant: by sweeping any representing measure μ to ∂K_1 we obtain a measure $\geq \mu_{\partial K_1}$ which represents z on $R(K_1)$, so that $\mu_{\partial K_1} \leq C_1 \lambda_z^1$, λ_z^1 being strongly dominant for $R(K_1)$. Similarly $\mu_{\partial K_0} \leq C_0 \lambda_z^0$, and (since $\partial U = \partial U_0 \cup \partial U_1$) by 3.9 $\lambda_z^1 + \lambda_z^0 \ll \lambda_z$, so $\mu \ll \lambda_z$. But we have no really direct proof that λ_z is strongly dominant, and argue as follows.

As noted, the dimension n of $(\operatorname{Re} R(K))^\perp$ is one less than the number of components of ∂K ; if U_1, \dots, U_k are the (finitely many) components of K^0 which are multiply connected, and U_i is n_i -connected, then $\sum_{i=1}^k n_i = n$ as is easily seen. So if we can produce, for each i , an independent set of n_i bounded real elements of $L^1(\lambda_{z_i})$ ($z_i \in U_i$) orthogonal to $R(K)$ we shall necessarily have a basis for $(\operatorname{Re} R(K))^\perp$, and then $(\operatorname{Re} R(K))_{\lambda_z}^\perp$ will always be spanned by finitely many bounded elements of $L^1(\lambda_z)$. Calling these $f_1\lambda_z, \dots, f_k\lambda_z$ say, we have $(1 \pm \varepsilon f_j)\lambda_z \geq 0$ for some $\varepsilon > 0$ and all $j=1, \dots, k$. Thus λ_z (when not unique) can be realized as a relative interior point of the convex set M_z , hence is strongly dominant.

Such an independent set of n_i bounded real elements μ_1, \dots, μ_{n_i} of $L^1(\lambda_{z_i})$ is

known, the natural set given by Ahern and Sarason [AS 1,2] and Wermer [We 4]: as a functional on $C(\partial U_i)$, μ_j is given by

$$\mu_j(f) = \int_{\gamma_j} \frac{\partial \hat{f}}{\partial n} ds$$

where⁽¹³⁾ \hat{f} is the continuous harmonic extension to U_i^- of f and $\gamma_1, \dots, \gamma_{n_i}$ is a homology basis of smooth curves in U .

For the final assertion we need only note that for any of our U_i no component of ∂U_i separates the remainder, so that exactly the argument of 3.4(g) applies to yield a basis of the prescribed special kind. This completes our proof of 3.13.

The preceding proof applies equally well to show the following.

PROPOSITION 3.14. *Suppose $K = \bigcap_{i=0}^n K_i$, where the compact K_i have pairwise disjoint connected boundaries covering ∂K and $R(K_i)$ is dirichlet on ∂K_i , $i=0, 1, \dots, n$. Then the rational defect for K is n , components of K^0 form the nontrivial parts, harmonic measures are strongly dominant, etc., as in 3.13.*

The methods developed so far yield a variety of K with infinitely many complementary components for which rational approximation holds. There are of course trivial examples; as earlier 3.4(d) and Kreĭn-Milman reduce showing $R(K) = A(K)$ to showing it for each component, so one trivial instance is that in which $S \setminus F$ has finitely many components for each component F of K . Less trivially, $R(K) = A(K)$ if ∂K is covered by the boundaries of finitely many holes: for then in the notation of 3.6, $K = K_0^*$ for a K^* to which 3.13 applies, so one concludes from $R(K)^\perp = \sum_\Delta R(K^*)^\perp$ that the rational defect for K is at most the sum of the connectivities of the multiply connected components of K^0 , and that is at least the analytic defect, as in 3.13, so both coincide. But we can in fact build much more complicated examples.

LEMMA 3.15. *Let $K = K_\infty \cap (\bigcap_{n=1}^\infty K_n)$, where $\{K_n\}$ is a decreasing sequence of compact connected sets in C and $\partial K_n \subset \partial K_{n+1}$ (so the components of $S \setminus K_n$ form a subset of those of $S \setminus K_{n+1}$, hence of $S \setminus K$), while $\partial K_n \cap \partial K_\infty = \emptyset$ for all n . Let E be the set of all z in ∂K with the property that each of its neighborhoods meets infinitely many of the sets $\partial K_{n+1} \setminus \partial K_n$. Suppose*

(a) $\partial K = \bigcup_{n=1}^\infty \partial K_n \cup \partial K_\infty \cup E$, and is connected.

⁽¹³⁾ The integral (where ds is the element of arc length) gives the period of the conjugate harmonic function, hence vanishes when that function is single valued, so μ_j is orthogonal to $\text{Re } R(K)$; that μ_1, \dots, μ_{n_i} are independent is easily seen and that μ_j is a bounded functional is a well-known consequence of the smoothness of the Green's function when U_i is bounded by analytic curves [Ne], and if $\tilde{U}_i \subset U_i$ contains our $\gamma_1, \dots, \gamma_{n_i}$ and is bounded by analytic curves we may obtain our μ_j as swept versions of those we obtain for \tilde{U}_i . This suffices to provide the representation of μ_j as a bounded function in $L^1(\lambda_{\tilde{U}_i})$ if \tilde{U}_i is relatively compact in U_i . (Also, if each γ_i is chosen to enclose precisely one component of ∂U_i , $\{\mu_j\}$ has the final property stated in 3.13, as is easily seen.)

(b) $R(K_n)$ is dirichlet on ∂K_n , $n=1, 2, \dots$ (and for $n=\infty$ as well if $K_\infty \neq S$), and each boundary is connected.

(c) E is a peak interpolation set⁽¹⁴⁾.

(d) Any two components of K^0 lie in distinct Gleason parts for $R(K^*)$ for some $K^* \supset K$.

(e) Each component of K^0 is nicely connected.
Then $R(K)$ is dirichlet on ∂K .

As an example of the application of 3.15, consider removing the interior of an infinitely branching tree of tangent closed discs from an annulus R as follows: starting with a root disc tangent to the outer bounding circle, remove a sequence of open discs proceeding upwards, branching to form a Y , the branches again branching etc., with the top of the tree moving towards (and accumulating on) the inner bounding circle of R . Suppose the leftmost route along the branches ("the left branch") leads to a single point z_0 of the inner bounding circle of R .

Now 3.15 does not apply directly: (d) will be hard to verify a priori, while (c) need not hold. But we need not apply 3.15 directly; the set K we obtain is the intersection of a sequence $\{K_\nu\}$, where K_ν has all but "the left branch" pruned, i.e., where all branches end ν discs from the root disc except for our path of discs leading to z_0 . So (cf. 3.1) we need only see each K_ν has $R(K_\nu)$ dirichlet, and 3.15 easily applies to $K=K_\nu$: $E=\{z_0\}$, and if two distinct components of K^0 appear, one is bounded by finitely many holes so a set K^* (to which 3.13 applies) is available as in (d).

Now suppose the branches do not accumulate at all points of the inner boundary of R , and let U be the union of the bounded components of $S \setminus K$. Choose a small replica U_1 of U lying in one component of K^0 with $\partial U_1 \cap \partial U$ a single point z_1 of the inner boundary of R . Taking $K_0 = S \setminus U_1$, $K_1 = K$ it is easy to see $\text{Re } (R(K_0 \cap K_1))^\perp$ is at most one-dimensional (removing a small neighborhood of z_1 from U_1 yields an altered K_0 with $\partial K_0 \cap \partial K = \emptyset$, so 3.2 applies; now apply 3.1). Thus 3.12 shows $R(K_0 \cap K_1) = R(K \setminus U_1)$ is dirichlet on $\partial(K \setminus U_1)$. Evidently we can now grow our "tree" in R using replicas of U instead of discs.

Proof of 3.15. Since $R(K)|_E = C(E)$, each $z \in E$ is a peak point of $R(K)|_E$, and since E is a peak set, z is a peak point of $R(K)$ by a result of Bishop [B 2]. Thus ∂K is the minimal boundary of $\bar{R}(K)$ by (a) and (b).

So each nontrivial part lies in K^0 and since a part for $R(K)$ is contained in one part for $R(K^*)$ when $K^* \supset K$, (d) implies the components U of K^0 are the nontrivial parts.

Now if $z \in U$ and $\mu \in M_z$ we have $\mu E = 0$ since if f peaks on E , $|f(z)| < 1$, so $\mu E = \lim \mu(f^n) = 0$. Thus we can see λ_z is dominant by showing $\mu \ll \lambda_z$ on $\partial K \setminus E$, hence on any compact subset F of $\partial K \setminus E$. Choose an open neighborhood V of F

⁽¹⁴⁾ That is, E is a peak set for which $R(K)|_E = C(E)$.

disjoint from E ; by compactness we have an n for which $F \subset \partial K_n \cup \partial K_\infty$ and $V \cap (\partial K_{k+1} \setminus \partial K_k) = \emptyset$ for $k \geq n$, so $V \cap (\partial K_{n+m} \setminus \partial K_n) = \emptyset$, $m \geq 1$. In (almost) the notation of 3.9, we set $K_0^* = K_n \cap K_\infty$ (so $S \setminus K_0^*$ consists of the components of $S \setminus K$ lying outside K_0^*), and take K_1^* as the complement of the union of the remaining components of $S \setminus K$, so $K = K_0^* \cap K_1^*$, $\partial K = \partial K_0^* \cup \partial K_1^* = \partial_0 \cup \partial_1$. Now sweeping μ to $\partial_0 = \partial K_0^*$ yields a measure $\geq \mu_{\partial_0}$ which represents z on $R(K_0^*)$, hence is bounded by $c\lambda_z^0$ since λ_z^0 is strongly dominant for $R(K_0^*) = R(K_n \cap K_\infty)$ by 3.14. So by 3.9 $\mu \ll \lambda_z^0 \ll \lambda^*$ on $\partial K_0^* \setminus (\partial_0 \cap \partial_1)$, in particular on F which lies in this set since $V \cap \partial_1 = \emptyset$. Since we have seen that the components of K^0 are parts, μ is singular with respect to λ_w , $w \notin U$, so $\mu \ll \lambda^*$ on F implies $\mu \ll \lambda_z$ on F and λ_z is dominant.

Now we know ∂K is the minimal boundary for $R(K)$, the components of K^0 are nicely connected and parts, and harmonic measures are dominant, so if $(\operatorname{Re} R(K))_\lambda^\perp$ is at most one-dimensional for $\lambda = \lambda_z$, $z \in K^0$, then by 3.10(b) representing measures are unique, which implies our result as we saw in 3.11. But $(\operatorname{Re} R(K))_\lambda^\perp$ is carried by $\partial K_\infty \cup \bigcup_{n=1}^\infty \partial K_n$ since $\lambda E = 0$, and if that space is of dimension greater than one, the same is true for the space N of real measures orthogonal to $R(K_n \cap K_\infty)$ we obtain by sweeping our measures to $\partial K_\infty \cup \partial K_n = \partial(K_\infty \cap K_n)$ for some n by 3.1; but since $\partial K_\infty \cap \partial K_n = \emptyset$, 3.2 implies N is at most one-dimensional, so that $(\operatorname{Re} R(K))_\lambda^\perp$ is also, and our proof is complete.

It should be noted how our last two results combine to yield further examples where $R(K) = A(K)$. These all have only finitely many multiply connected components of K^0 , and the rational defect is finite. To arrive at examples with infinite rational defect we need

LEMMA 3.16. *Suppose z_0 lies in a component of K^0 which is a part for $R(K)$, and λ is the harmonic measure for z_0 . Then in $L^2(\lambda)$, $E = H^2(A(K)) \ominus H^2(R(K))$ is finite dimensional only if $E = \{0\}$.*

Proof. Suppose $E \neq \{0\}$ and let g_1, \dots, g_n be an orthonormal basis for E . Since $H^2(A(K))$ is an $R(K)$ -module (which, as the reader will note, is almost the only property of $H^2(A(K))$ we shall use) for $g \in E$ and $r \in R(K)$ $rg = g' + r'$ with $g' \in E$ and $r' \in H^2(R(K))$ unique, so $T_r g = g'$ defines a representation T of $R(K)$ on E . Since the representation is finite dimensional and $R(K)$ abelian we have a common eigenvector which we can take as g_1 . Thus $T_r g_1 = \phi(r) \cdot g_1$, or $rg_1 = \phi(r)g_1 + r'$, where $\phi(r) \in \mathbb{C}$ and $r' \in H^2(R(K))$. Trivially ϕ is a multiplicative linear functional, 1 at 1, and so corresponds to evaluation at $z_1 \in K$. And

$$\begin{aligned} |g_1|^2 \lambda(r) &= \bar{g}_1 \lambda(r g_1) = \bar{g}_1 \lambda(\phi(r) g_1 + r') \\ &= \phi(r) \lambda(|g_1|^2) + \bar{g}_1 \lambda(r') = \phi(r) \cdot 1 + 0 = r(z_1) \end{aligned}$$

since g_1 is orthogonal to $H^2(R(K))$ in $L^2(\lambda)$. This shows z_1 lies in the same part as z_0 , hence in the same component of K^0 , and $\lambda_1 = \lambda_{z_1}$ is boundedly equivalent to λ .

In particular then λ_1 yields a continuous functional⁽¹⁵⁾ on $L^2(\lambda)$, and one which is multiplicative on the $L^2(\lambda)$ - (or $L^2(\lambda_1)$ -) closure of $A(K)$. So since $\phi(r)=z_1$ for $r(z)\equiv z$, $zg_1=z_1g_1+r'$, $r'\in H^2(R(K))$, which implies

$$\lambda_1(r') = \lambda_1(z-z_1) \cdot \lambda_1(g_1) = 0 \cdot \lambda_1(g_1) = 0.$$

But $r \rightarrow \lambda_1(r)=r(z_1)$ is continuous on $R(K)$ in the $L^2(\lambda)$ norm, so that if r_v in $R(K)$ converges to r' in $L^2(\lambda)$, $r_v(z_1) \rightarrow \lambda_1(r')=0$, and we may just as well take $r_v(z_1)=0$. Thus $r_v/(z-z_1) \in R(K)$, $r_v/(z-z_1) \rightarrow r'/(z-z_1)=g_1$ in $L^2(\lambda)$, and we find $g_1 \in H^2(R(K))$, our contradiction⁽¹⁶⁾.

The fact that the proof of 3.16 applies with $H^2(A(K))$ any $R(K)$ -module on which λ_{z_1} is multiplicative allows us to draw the following consequence, which even here is weaker than the result obtained by Ahern and Sarason [AS 1, 14.5] for hypo-dirichlet algebras when $R(K)$ is such an algebra.

COROLLARY 3.17. *Suppose z_0 lies in a component U of K^0 which is a part for $R(K)$, the harmonic measure for z_0 is $\lambda=\lambda_{z_0}$, and in $L^2(\lambda)$ the subspace orthogonal to $R(K)+\bar{R}(K)$ is finite-dimensional. Then $H^\infty(R(K), \lambda)$ is maximal among the subalgebras of $L^\infty(\lambda)$ on which λ_z is multiplicative for every z in U .*

Indeed if $B \supset H^\infty(R(K))$ is such an algebra then since λ is multiplicative on $B \supset R(K)$, B is orthogonal in $L^2(\lambda)$ to $\bar{R}(K)_{z_0}$ and thus, if $H^2(B)$ is the closure of B in $L^2(\lambda)$, $E=H^2(B) \ominus H^2(R(K))$ is orthogonal to $R(K)+\bar{R}(K)_{z_0}=R(K)+\bar{R}(K)$, hence finite-dimensional. By the proof of 3.16 we conclude that $E=\{0\}$, so each $f \in B$ lies in $H^2(R(K)) \cap L^\infty = H^\infty(R(K))$. Except for the possibility that no such algebra B may exist, we are done; but as noted after 2.14, $B=H^\infty(R(K))$ is one.

We have noted 3.17 since it suffices to yield the consequence of [AS 1, 14.5] Ahern and Sarason obtain in [AS 2, Theorem 2] by exactly their proof.

If U is an n -connected component of K^0 it is the conformal image of a domain D bounded by $n+1$ disjoint circles [Ne]; if the corresponding map ρ has nontangential boundary values which yield a 1-1 map ρ^* on almost all of ∂D (relative to harmonic measure μ for any point w in D), as in [AS 2, §3], we shall say U is *nicely n -connected*. This holds in particular if $S \setminus K$ has finitely many components [AS 2, §3], and allows

⁽¹⁵⁾ If g_1 were bounded, $r \rightarrow |g_1|^2 \lambda(r)=r(z_1)$ would extend to a continuous functional on $L^2(\lambda)$; and so (without assuming λ is the harmonic measure and U a part, but using $|g_1|^2 \lambda$ in place of λ) the proof leads again to a contradiction. Thus if λ is any representing measure on $R(K)$ and $E=H^2(A(K), \lambda) \ominus H^2(R(K), \lambda)$ is finite dimensional and consists of bounded functions, $E=\{0\}$.

⁽¹⁶⁾ A similar argument shows that whenever ∂K is the minimal boundary for $R(K)$, $R(K)$ cannot have finite positive defect in $A(K)$. (Take a finite-dimensional complementary subspace E of $R(K)$ in $A(K)$; one again gets a representation T and a common eigenvector g in E , and $rg=r(z_1)g+r'$, $r' \in R(K)$. Choosing a measure μ on ∂K orthogonal to $R(K)$ with $\mu(g)=1$, $\mu(rg)=r(z_1)$; since $z \in \partial K$ is a peak point, $\mu\{z\}=0$ for each $z \in \partial K$, so $z_1 \notin \partial K$. Thus $(z-z_1)g=r'$ implies $r'(z_1)=0$ and so $g \in R(K)$.)

one to obtain the measure space associated with ∂U and $\lambda_{\rho(w)}$ as the transported version of that associated with ∂D and μ . And in this situation as in [AS 2, §3] we can transplant the boundary value functions of $\mathfrak{S}(D)$, the space of bounded analytic functions on D , to yield an algebra $B \supset H^\infty(R(K), \lambda_{\rho(w)})$ on which $\lambda_{\rho(w)}$ is multiplicative; thus by exactly the proof of [AS 2, Theorem 2] using 3.17 in place of Ahern and Sarason's Lemma 4, we obtain

COROLLARY 3.18 [AS 2]. *Suppose z_0 lies in a nicely n -connected component U of K^0 which is a part for $R(K)$, and in $L^2(\lambda_{z_0})$ the subspace orthogonal to $\text{Re } R(K)$ is finite dimensional. Then if $\mathfrak{S}(U)$ is the (sup-normed) space of bounded analytic functions on U , $H^\infty(R(K), \lambda_{z_0})$ is isometrically isomorphic to $\mathfrak{S}(U)$ under the map $f \rightarrow \hat{f}$, where $\hat{f}(z) = \lambda_z(f)$, $f \in H^\infty(R(K), \lambda_{z_0})$.*

We proceed to our main consequence of 3.16.

THEOREM 3.19. *Suppose: (1) the minimal boundary for $R(K)$ covers all of ∂K except for a set of plane measure zero (i.e., that $R(\partial K) = C(\partial K)$ by Bishop's theorem [B 2]) and (2) for each component U of K^0 there is a compact $K_U \supset K$ having U as a component of K_U^0 and a part for $R(K_U)$, while the harmonic measure λ_z for $z \in U$ is strongly dominant for $R(K_U)$ and $(\text{Re } R(K_U))_{\lambda_z}^\perp$ is finite dimensional.*

Then $R(K) = A(K)$, the nontrivial parts are the components of K^0 , and harmonic measures are strongly dominant.

In particular, if (1) holds and each component U of K^0 is bounded by finitely many holes, $R(K) = A(K)$ since K_U can be taken as the complement in S of the union of those holes, and 3.13 applies to K_U .

To proceed to the proof of 3.19, note first that U , as a part for $R(K_U) \subset R(K)$, is a part for $R(K)$. So K^0 is a union of parts and each $z \in \partial K$ lies in a part contained in ∂K . By a recent result⁽¹⁷⁾ of Wilken [Wi], this and (1) imply z is itself a part and in fact a peak point for $R(K)$. So ∂K is the minimal boundary for $R(K)$, and the nontrivial parts are precisely the components of K^0 .

Let U be a component of K^0 , $z \in U$. Since U is also a component of K_U the harmonic measure λ_z relative to either K or K_U is the same. Moreover $\mu \in M_z(R(K))$, when swept to a measure μ' on K_U representing z on $R(K_U)$ has $\mu' \leq C_z \cdot \lambda_z$ since λ_z is strongly dominant for $R(K_U)$. But evidently $\mu = \mu'$ (so that $M_z(R(K)) \subset M_z(R(K_U))$ and λ_z is strongly dominant for $R(K)$): for we have a Borel set $E \subset \partial U$ carrying λ_z (and so μ') and of measure zero⁽¹⁸⁾ for $\lambda_w^{K_U}$, $w \in K_U \setminus U^-$, so that

$$\mu' = \mu_{\partial U} + \int_{\partial K \setminus U^-} \lambda_w^{K_U} \mu(dw)$$

implies $1 = \mu'(E) = \mu_{\partial U}(E) + 0$, and $\mu' = \mu_{\partial U} = \mu$.

⁽¹⁷⁾ Which states that $z \in K$ is either a peak point for $R(K)$ or lies in a part with positive plane measure (for any K).

⁽¹⁸⁾ By Wilken's result there are only countably many non-peak-point-parts for $R(K_U)$ so we only need to insure E' supports countably many measures singular with respect to λ_z ; for peak points w in $K_U \setminus U^-$, $\lambda_w^{K_U}(E) = \delta_w(E) = 0$ automatically.

By 3.6 we have no completely singular orthogonal measures, and to prove $A(K) = R(K)$ we need only show $A(K)_{\lambda_z}^\perp = R(K)_{\lambda_z}^\perp$ for all z in K^0 . Taking $z \in U$ as above, by 2.12 we need only see the $L^2(\lambda_z)$ -closures of $A(K)$ and $R(K)$ coincide since λ_z is strongly dominant.

But $E = H^2(A(K)) \ominus H^2(R(K))$ lies in the subspace of $L^2(\lambda_z)$ orthogonal to $\operatorname{Re} R(K)$, so $E \cdot \lambda \subset (\operatorname{Re} R(K))_{\lambda_z}^\perp \subset (\operatorname{Re} R(K_U))_{\lambda_z}^\perp$ which is finite dimensional, and 3.16 applies to show $E = \{0\}$, completing our proof.

Some final comments are in order on the components $R(K)_\lambda^\perp$ of the decomposition (2.3) we obtain for those K which we can construct via 3.14, 3.15, 3.19. When K is the minimal boundary, components are parts and harmonic measures are dominant, the final assertion of 3.6 shows that each component $R(K)_\lambda^\perp$ is precisely $R(U^-)^\perp$ for the corresponding component U of K^0 . And whenever U is a nicely n -connected component of K^0 which is a part, while the harmonic measure λ_z of one of its points is strongly dominant then $R(K)_{\lambda_z}^\perp$ is merely the transported version of a space $R(B)^\perp$, where B is the closure of a domain bounded by finitely many disjoint circles conformally equivalent to U .

Indeed if ρ is the conformal map of B^0 onto U , then we know the measure space associated with ∂U and λ_z is that associated with ∂B and $\lambda_\rho^{B^0 - 1(z)} = \mu$ transported via the map ρ^* , and then $H^\infty(\mu)$ corresponds precisely to $H^\infty(\lambda_z)$ since $(H^\infty(\lambda_z) \circ \rho^*)^\wedge = H^\infty(\lambda_z)^\wedge \circ \rho = \mathfrak{F}(U) \circ \rho = \mathfrak{F}(B^0) = H^\infty(\mu)^\wedge$, as is easily verified (using 3.18 of course), where, as there, $\hat{f}(w) = \lambda_w(f)$, etc. Since $f \rightarrow \hat{f}$ is an isomorphism, $H^\infty(\lambda_z) \circ \rho^* = H^\infty(\mu)$. But the elements of $R(K)_{\lambda_z}^\perp$, or of $R(B)_\mu^\perp = R(B)^\perp$, are just those of $L^1(\lambda_z)$ or $L^1(\mu)$ orthogonal to $H^\infty(\lambda_z)$ or $H^\infty(\mu)$ (by 2.9), so $H^\infty(\lambda_z) \circ \rho^* = H^\infty(\mu)$ implies $\rho^* R(B)^\perp = R(K)_{\lambda_z}^\perp$. (In particular then, the real orthogonal measures are given by bounded functions in $L^1(\lambda_z)$.)

4. Interpolation as an application. Let K again be a compact plane set, and let $\lambda^* = \sum_{n=1}^\infty 2^{-n} \lambda_n$ where λ_n is the harmonic measure of a point in the n th component of K^0 as in 3.9. In those cases in which we have been able to show $R(K) = A(K)$ in the preceding section, we have also obtained considerable information on $A(K)^\perp$ (cf. 2.8, 2.10, 3.6); in particular it lies in $L^1(\lambda^*)$, and as an immediate consequence we know the closed subsets F of ∂K for which $A(K)$ interpolates $C(F)$: $A(K)|_F = C(F)$ iff $\lambda^* F = 0$. (Indeed if $\lambda^* F = 0$ (and $A(K)^\perp \subset L^1(\lambda^*)$) then $\mu_F = 0$ for each μ in $A(K)^\perp$, so our conclusion follows by [Gl 1, 4.8], which in fact shows F is a peak set as well. On the other hand if $\lambda_z F > 0$ for some z in some component U of K^0 then this inequality holds for all z in U . Choose $f \in C(F)$ which doesn't vanish at some $z_0 \in F \cap U^-$, but does vanish on a subset of F positive λ_z -measure (for some, hence all, z in U). If $g \in A(K)$ has $g|_F = f$ then $g(z) = \lambda_z(g) = 0$ for z in U by Jensen's inequality, so $g|_{U^-} = 0$ and $f(z_0) = 0$. Thus $\lambda^* F = 0$ if $A(K)|_F = C(F)$.)

What we want to point out in the present section is that the subsets of K (rather than just ∂K) which are sets of interpolation can essentially be determined in the same setting.

Let $\mathfrak{H}(U)$ be the sup-normed space of bounded analytic functions on the open set $U \subset S$, and for $F \subset U$ say $\mathfrak{H}(U)$ *interpolates* $C(F)$ with bound M if, for each f in $C(F)$, there is a g in $\mathfrak{H}(U)$ with $g|_F = f$, $\|g\| \leq M \|f\|$. (Of course F must be given by a sequence with no cluster points in U .)

THEOREM 4.1. *Suppose ∂K is the minimal boundary for $R(K)$; each component U of K^0 is nicely n -connected and a part for $R(K)$ with $\lambda = \lambda_z$ ($z \in U$) strongly dominant; finally that the subspace of $L^2(\lambda)$ orthogonal to $\text{Re } R(K)$ is finite dimensional.*

Let F be a closed subset of K . Then⁽¹⁹⁾ $A(K) = R(K)$ interpolates $C(F)$ iff

- (a) $\lambda^*(F \cap \partial K) = 0$, and
- (b) *there is a constant M for which $\mathfrak{H}(U)$ interpolates $C(F \cap U)$ with bound M for each component U of K^0 .*

(Note that the second hypothesis holds if each U is bounded by finitely many components of $S \setminus K$.) Thus interpolation by $A(K)$ reduces, on K^0 , to interpolation by bounded analytic functions (with some uniformity as we pass among the components of K^0), exactly as in the case of the unit disc [H 2]. If U is an n -connected component of K^0 , and we divide $F \cap U$ into $n+1$ sequences F_i , $i=0, \dots, n$, with F_i clustering only on the i th component ∂_i of U , and U_i is the simply connected component of $S \setminus \partial_i$ containing U , then Stout [S] has shown that interpolation of $C(F \cap U)$ by $\mathfrak{H}(U)$ is precisely equivalent to interpolation of $C(F_i)$ by $\mathfrak{H}(U_i)$ for all i , and this of course can be stated in terms of Carleson's criterion [H 2]. At least when only finitely many components of K^0 are multiply connected (so they can be neglected in showing M exists) we can thus easily express (b) entirely in terms of a bound on the compositions of certain Blaschke products and Riemann mappings.

Let $K^0 \cap F = \{z_n\} = \{z_1, z_2, \dots\}$. For the proof of 4.1 we must depart from the convention of the preceding section and consider measures on $X = \partial K \cup F$ orthogonal to $A(K)$, (respectively $R(K)$); however $A(K)^\perp$ will still denote the measures on ∂K orthogonal to $A(K)$. In order to identify the measures on $X = \partial K \cup F$ orthogonal to $A(K)$ we need the following simple fact: Let $\nu_n = \delta_{z_n} - \lambda_{z_n}$, so $\nu_n \perp A(K)$. Then the set of measures on $X = \partial K \cup F = \partial K \cup \{z_n\}$ orthogonal to $A(K)$ is

$$(4.1) \quad \left\{ \sum_{n=1}^{\infty} c_n \nu_n + \mu : c \in l_1, \mu \in A(K)^\perp \right\}.$$

Indeed (4.1) is clearly orthogonal to $A(K)|_X$ while if ν is a measure on X orthogonal to $A(K)$ then $\nu_{\{z_1, z_2, \dots\}} = \sum c_n \delta_{z_n}$ with $c \in l_1$, so that $-\sum c_n \nu_n + \nu = \sum c_n \lambda_{z_n} + \nu_{\partial K} = \mu$ is a measure on ∂K orthogonal to $A(K)$, so in $A(K)^\perp$, and $\nu = \sum c_n \nu_n + \mu$ as desired.

We shall now prove 4.1. Suppose (a) and (b) hold. We shall show $A(K)$ interpolates $C(F)$ by showing there is a constant C for which

$$(4.2) \quad \|\nu_F\| \leq C \|\nu_{X \setminus F}\|,$$

⁽¹⁹⁾ $A(K)_\lambda^\perp = R(K)_\lambda^\perp$ by 3.16 and 2.12, so rational approximation follows.

for all ν in (4.1), which is sufficient by [Gl 1, 3.2]. But $\nu = \sum_n c_n \nu_n + \mu = \sum_n c_n \nu_n + \sum_k \mu_{\lambda_k}$ where λ_k is harmonic measure of a point in the k th component U_k of K^0 as earlier and $\mu_{\lambda_k} \perp A(K)$ by 2.6; collecting together μ_{λ_k} and all terms of $\sum c_n \nu_n$ corresponding to those z_n lying in U_k , we express ν as a norm convergent sum of mutually singular elements of (4.1), each summand being a measure carried by the closure of one component of K^0 . So it suffices to prove we have a C satisfying (4.2) for all ν carried by the closure of one component U of K^0 , with C independent of U .

Let λ be the harmonic measure of some z in U ; λ is strongly dominant for $A(K)$ by hypothesis. For $\nu = \sum c_n \nu_n + \mu$, with $\mu \ll \lambda$ and $c_n = 0$ if $z_n \notin U$, we have $\nu_F = \sum c_n \nu_{nF} = \sum c_n \delta_{z_n}$ since $\lambda F = 0$. For $\varepsilon > 0$ choose N so large that $\|\nu_F\| - \varepsilon/2M < \sum_{n=1}^N |c_n|$ (and therefore $\sum_{n=N+1}^{\infty} |c_n| < \varepsilon/2M$ as well) where M , our constant in (b), is at least 1. Set

$$f(z_n) = \overline{\operatorname{sgn} c_n};$$

by (b) we have a g in $\mathfrak{S}(U)$ with $g=f$ on $U \cap \{z_n\}$ and $\|g\|_{\infty} \leq M\|f\|_{\infty} = M$. By 3.18 we have an h in $H^{\infty}(A(K), \lambda)$ for which $\hat{h}=g$ on U and $\|h\|_{\infty} = \|g\|_{\infty} \leq M$. So since λ is strongly dominant, by 2.9 we have f_k in $A(K)$ of norm $\leq M$ with $f_k \rightarrow h$ a.e. λ , so that $\hat{f}_k \rightarrow \hat{h}=g$ pointwise on $\{z_n\} \cap U$, and

$$\hat{f}_k(z_n) \rightarrow \overline{\operatorname{sgn} c_n}.$$

Thus

$$\begin{aligned} \|\nu_F\| - \varepsilon &\leq \|\nu_F\| - \frac{\varepsilon}{2M} - \frac{\varepsilon}{2M} \cdot M \leq \sum_{n=1}^N |c_n| - \sum_{n=N+1}^{\infty} |c_n| \cdot M \\ &\leq |\nu_F(f_n)| + \varepsilon \end{aligned}$$

for n sufficiently large, and since $\nu(f_n)=0$, $\|\nu_F\| - \varepsilon \leq |\nu_{X \setminus F}(f_n)| + \varepsilon \leq M\|\nu_{X \setminus F}\| + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, (4.2) follows with $C=M$ independent of U .

To see the converse is easy. Indeed (a) holds by the argument used in the second paragraph of this section, while $A(K)|F = C(F)$ implies that $g \rightarrow g|F$ maps $A(K)$ onto $C(F)$, so by the open mapping theorem we have a constant M for which, for each $f \in C(F)$ we have a g in $A(K)$ with $g=f$ on F and $\|g\| \leq M\|f\|$. Thus if $f \in C(F \cap U)$ and we set $f_N=f$ on $U \cap \{z_1, \dots, z_N\}$ and $=0$ elsewhere on $\{z_n\}$ we have $g_N \in A(K)$ of norm $\leq M\|f\|$ and $g_N=f$ on $U \cap \{z_1, \dots, z_N\}$. Since g_N is a normal family in $\mathfrak{S}(U)$ we of course obtain $g \in \mathfrak{S}(U)$ with the required properties.

5. Approximation within $R(K)$. In certain situations the results of §§2 and 3 can be used to show the density of some subalgebras of $R(K)$. For example suppose each z in K has a strongly dominant representing measure for $R(K)$, while A is a closed subalgebra of $R(K)$ with K as its spectrum and $(\operatorname{Re} A)^{\perp} = (\operatorname{Re} R(K))^{\perp}$. Then A and $R(K)$ share the same set of multiplicative measures (if μ is a multiplicative on A and represents z , $\mu - \lambda_z \in (\operatorname{Re} A)^{\perp}$ is orthogonal to $R(K)$, so μ is multiplicative on $R(K)$), and thus have the same Gleason parts. So if A has no completely singular orthogonal measures $A=R(K)$ by 2.12 and 2.8.

This section is devoted to an instance in which our earlier arguments eliminate the completely singular measures. The result is an awkward relative of a result of Bishop [B 1].

THEOREM 5.1. *Suppose strongly dominant measures exist for $R(K)$, harmonic measures are dominant, and ∂K has plane measure zero⁽²⁰⁾. Suppose A is a closed, inverse closed subalgebra of $R(K)$ with the same real orthogonal measures as $R(K)$ on ∂K , and contains a subset \mathcal{G} whose elements do not vanish on ∂K and satisfy the following:*

- (a) $1 \in \mathcal{G}$ and $g - c \in \mathcal{G}$ if $g \in \mathcal{G}$ and $c \in C \setminus g(\partial K)$.
- (b) $g \in \mathcal{G}$ is analytic near K .
- (c) \mathcal{G} separates ∂K .
- (d) Let $g \in \mathcal{G}$. Then for each $z \in \partial K$ there is a closed neighborhood V of z in K and an $f \in C + gA$ for which: (i) harmonic measures are dominant for $R(f(K))$ (so $\partial f(K)$ is the minimal boundary, in particular); (ii) f maps V homeomorphically with $V = f^{-1}f(V)$.

Then $A = R(K)$.

For the purposes of our proof we let λ_K^* denote our measure $\lambda^* = \sum 2^{-n} \lambda_n$, where λ_n is the harmonic measure of some point in the n th component of K^0 ; λ_V^* denotes the corresponding measure for V . Let $B = C + gA$ as in (d), and let μ be a probability measure on ∂K , multiplicative on B , and not a point mass. We shall show $\mu \ll \lambda_K^*$.

Let z lie in the closed support of μ , and choose V and f as in (d). Shrinking V (so f becomes an open homeomorphism of a neighborhood of V in K onto a neighborhood of $f(V)$ in $f(K)$) we can assume $V = K \cap W$, with W a closed disc about z , and that f maps $V \cap \partial K$ into $\partial f(K)$. With $V = K \cap W$, all points of ∂V are regular [T], and the same is true for $\partial f(V)$: for f is conformal on $V \setminus (V \cap \partial K)$, and $V \cap \partial K$ maps into $\partial f(K)$ which is the minimal boundary for $R(f(K))$ so that barriers exist for all points in $\partial f(V)$.

We need the fact that on the relatively open subset $E = V \cap \partial K \setminus \partial W = \partial V \setminus \partial W$ of ∂V we have $\lambda_K^* \ll \lambda_V^*$ (and the corresponding fact for $f(K)$, $f(V)$ and $f(E) = \partial f(V) \setminus f(\partial W)$), and this follows from the argument used to prove 3.9, which we briefly recapitulate (omitting the passage from continuous to discontinuous boundary values, as given in 3.9).

If F is a compact subset of E and $\lambda_V^* F = 0$, setting

$$\begin{aligned} f &= 0 \text{ on } E, \\ &= \lambda_z F \text{ on } V \setminus E, \end{aligned}$$

we have $f \in C(\partial V)$, and the harmonic function $w \rightarrow \lambda_w F - \hat{f}(w)$ on V^0 (where \hat{f} is the continuous harmonic extension of f to V) has boundary values $= \chi_F$ on E ,

⁽²⁰⁾ This condition and (b) below are chosen to insure that $R(g(\partial K)) = C(g(\partial K))$ for any $g \in \mathcal{G}$; any other conditions insuring this would do as well.

and $=0$ on $\partial V \setminus E$, and so coincides with $w \rightarrow \lambda_w^V(F)$, which vanishes on V^0 since $\lambda_w^* F = 0$. Thus $\lambda_w F = \hat{f}(w)$ on V^0 , and since $\hat{f} \rightarrow 0$ as we approach F in V^0 , hence in K^0 , while $\lambda_w F \rightarrow 0$ as we approach ∂K^0 in K^0 away from F , we see the harmonic function $w \rightarrow \lambda_w F$ vanishes on K^0 , so that $\lambda_K^* F = 0$ as desired. Similarly $\lambda_{f(K)}^* \ll \lambda_{f(V)}^*$ on $\partial f(V) \setminus f(\partial W)$.

Now let f^* denote the map of measures on ∂K into measures on $f(K)$ dual to the map $g \rightarrow g \circ f$ of $C(f(K))$ into $C(\partial K)$. Then since A is inverse closed, for any rational function g without poles in $f(K)$ we have $g \circ f \in A$ (if p is a polynomial without zeroes on $f(K)$, $p \circ f \in A$ does not vanish on K , so $1/(p \circ f) = (1/p) \circ f \in A$). So $g \circ f \in A$ for $g \in R(f(K))$, and thus $g \rightarrow f^* \mu(g) = \mu(g \circ f)$ is multiplicative on $R(f(K))$. Moreover $f^* \mu$, which has $f(z)$ in its closed support, is not a point mass unless it represents $f(z)$, and then μ would necessarily be δ_z since f is a homeomorphism on V and $V = f^{-1}f(V)$.

Thus, swept to $\partial f(K)$, $f^* \mu$ yields a non-point-mass multiplicative measure on $R(f(K))$ which is $\geq (f^* \mu)_{f(V)} = f^*(\mu_V)$ since the support of μ_V , which lies in $V \cap \partial K$, maps into $\partial f(K)$. Since the swept measure must represent a point in $f(K)^0$ by (d), $(f^* \mu)_{f(V)} \ll \lambda_{f(K)}^*$, again by (d). As noted $\lambda_{f(K)}^* \ll \lambda_{f(V)}^*$ on $\partial f(V) \setminus f(\partial W)$, hence on $f(K \cap W^0)$ so that $f^* \mu \ll \lambda_{f(V)}^*$ on $f(K \cap W^0)$, which implies $\mu \ll \lambda_K^*$ on $K \cap W^0$. But $\lambda_K^* \ll \lambda_K^*$ on $K \cap W^0$, as one sees by sweeping, so $\mu_{K \cap W^0} \ll \lambda_K^*$. Since z is an arbitrary point in the support of μ , $\mu \ll \lambda_K^*$.

In particular for the element 1 of B we see $B = C + 1A = A$ has any (nondiscrete) multiplicative measure $\mu \ll \lambda_K^*$, so μ represents a point in \mathfrak{M}_A lying in the same Gleason part as some z in K^0 . But $M_z(A) = M_z(R(K))$ since if $\nu \in M_z(A)$, $\nu - \lambda_z \in (\text{Re } A)^\perp = (\text{Re } R(K))^\perp$ implies $\nu \in M_z(R(K))$. Thus for each part P for A we have an element z of K in P with a strongly dominant representing measure λ , equivalent to λ_z since harmonic measures are dominant for $R(K)$ by hypothesis. Since $(\text{Re } A)^\perp = (\text{Re } R(K))^\perp$, 2.12 shows $A_\lambda^\perp = R(K)_\lambda^\perp$, and thus in the decomposition of A^\perp and of $R(K)^\perp$, given by 2.8, A^\perp and $R(K)^\perp$ can only differ in the completely singular components: so to see $A = R(K)$ it suffices to show $S = 0$, where S is the set of completely singular measures orthogonal to A . (The corresponding measures for $R(K)$ are included in S .)

Now let $B_0 = \{g \in C(\partial K) : g \perp S\}$. B_0 is a (closed) subalgebra of $C(\partial K)$ since $g \in B_0$, $\sigma \in S$ imply $g \perp f\sigma$ for $f \in A$, or $g\sigma \perp f$; so $g\sigma \perp A$, and being singular, $g\sigma \in S$. Thus $hg\sigma \in S$ for $h \in B_0$, and $\sigma(hg) = gh\sigma(1) = 0$, so $hg \in B_0$, and B_0 is an algebra. Of course A is a subalgebra, and we shall next see each $g \in \mathcal{G}$ has its spectrum in B_0 precisely $g(\partial K)$.

The spectrum of g in A is contained in $g(K)$ as we already know, since A is inverse closed. So the spectrum of g in B_0 lies in $g(K)$. For $t = g(z) \in g(K)$, $z \in K$, $t \notin g(\partial K)$, we have $g_1 = g - t$ an invertible in $C(\partial K)$ which lies in \mathcal{G} . By our earlier argument we know each measure μ which is multiplicative on $B = C + g_1 A$ and not a point-mass is $\ll \lambda_K^*$. Thus dominant measures exist for B .

Now let $\sigma \in S$, so that $\sigma \perp B$, and, since $|\sigma|$ and λ_K^* are mutually singular, σ is a

completely singular orthogonal measure for B . For our $z \in K$ with $g(z)=t$ we have $g_1(z)=0$, so

$$(1/g_1)\sigma - \sigma(1/g_1) \cdot \lambda_z$$

is orthogonal to $C + g_1 \cdot A = B$; by 2.6, $(1/g_1)\sigma$, its singular component relative to the dominant measure for z on B , is also orthogonal to B . So $\sigma(1/g_1) = (1/g_1)\sigma(1) = 0$. Since this holds for all $\sigma \in S$ we have $1/g_1 \in B_0$, so that the spectrum of $g \in \mathcal{G}$ in B_0 is $g(\partial K)$ as asserted. Hence $h \circ g \in B_0$ for each $h \in R(g(\partial K))$.

But $g \in \mathcal{G}$ is analytic near K and ∂K has plane measure zero, so that $g(\partial K)$ also has plane measure zero, and $R(g(\partial K)) = C(g(\partial K))$ by the well-known result of Hartogs and Rosenthal (which is contained in the familiar result of Bishop [B 2]), and thus $h \circ g B_0$ for each $h \in C(g(\partial K))$, which in particular says $\bar{g} \in B_0$. Since the elements of \mathcal{G} separate ∂K , we have $B_0 = C(\partial K)$ by Stone-Weierstrass, and $S=0$ as desired, completing our proof.

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